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N-Butterflies: Modeling Weak Morphisms of Strict N-Groups

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N-BUTTERFLIES: MODELING WEAK MORPHISMS OF STRICT N-GROUPS

By

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To my family

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NOTATIONS AND TERMINOLOGY

We will assume a basic knowledge of algebraic topology (see [17]) and homological algebra (see [29]). The reader should have an introductory understanding of category theory (see [22] or [4]) and model categories (see Appendix A). For a more detailed exposition, see [13]. A particularly important category will be the category of simplicial sets which is thoroughly discussed in [15].

The category of sets, categories, and simplicial sets will be denoted by **Set**, **Cat**, and **sSet**, respectively. Since many structures have higher generalizations, we will use the same notation for the higher counterparts. For example, ∞-categories will use the same notation as categories. Generally we have,

$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ for categories;

A, B, C, D, W, X, Y, Z for objects of categories;

f, g, h for morphisms;

F, G, H for functors.

If no confusion can arise, we will usually reserve G and H with their respective fonts for group objects. Multiplicative notation will be assumed unless explicitly stated. Also, the number 1 will be used broadly to denote the identity. For example, the identity of a group or the identity map. Lastly, we will denote the set of morphisms of a category $\mathcal{C}$ from an object x to an object y by $\mathcal{C}(x, y)$ instead of the usual *hom* set notation. All ambiguities should be clear from the context and any other notations will be noted as they arise naturally.

ABSTRACT

Butterflies are an algebraic model of the morphisms of the homotopy category of crossed modules and were originally introduced by Behrang Noohi. Crossed complexes are algebraic structures which generalize crossed modules. The following dissertation is concerned with adapting butterflies to the full subcategory of crossed complexes called reduced n-crossed complexes.

CHAPTER 1

INTRODUCTION

In many cases, studying topological spaces up to isomorphism is to rigid; instead, a usually more profitable notion of equality between two spaces is the existence of a homotopy equivalence. Although homotopy equivalence is weaker than isomorphism, it is well-known that homotopy is an equivalence relation; thereby, giving topologists a well-defined way to formally invert homotopy equivalence. In topology, sometimes an even weaker notion of equality is desired; in particular, when there is a continuous map between two spaces which induces isomorphisms of homotopy groups. We will call such maps weak equivalences. Unfortunately, the passage to inverting weak equivalences is not as clear since such morphisms are not necessarily symmetric.

Daniel Quillen realized that the usual notion of homotopy could be generalized so that a map is a homotopy equivalence if and only if it is a morphism which induces isomorphisms of homotopy groups [27]. Futhermore, the generalized notion of homotopy is an equivalence relation. Hence, these morphisms can be formally inverted by set-theoretic means. Using category theory, Quillen was able to generalize homotopy theory further to objects, not necessarily topological spaces, where a subclass of morphisms could be inverted. In particular, the morphisms of this subclass are called weak equivalences. Quillen derived the precise properties a category with a class of weak equivalences must satisfy to admit a well-defined notion of homotopy in which the weak equivalences can be inverted. For all intents and purposes, we say that such a category $\mathcal{M}$ exhibits a model structure. Moreover, a new category $Ho(\mathcal{M})$, called the homotopy category, can be constructed which has the same objects as $\mathcal{M}$ and the weak equivalences are isomorphisms. For objects $X, Y \in Ho(\mathcal{M})$, the set of morphism from X to Y in $Ho(\mathcal{M})$ is denoted by $[X, Y]_{Ho(\mathcal{M})}$. It is worth noting that a category could exhibit multiple model structures and thus, multiple homotopy categories. The category of topological spaces with the class of weak equivalences defined above form a model category $\mathbf{Top_Q}$ and a homotopy category $Ho(\mathbf{Top_Q})$.

The subcategory of $Ho(\mathbf{Top_Q})$ with objects having trivial homotopy groups above degree two is called the category of homotopy 2-types and denoted by $\mathbf{H_2Typ}$. A category with a model structure which is not the category of topological spaces is the category of crossed modules (see Chapter 2). Informally, the objects of this category are just group homomorphisms with an action which satisfies certain properties. A well-known fact is that the category of crossed modules, denoted by $\mathbf{xm}$, with the Moerdijk-Svensson model structure are algebraic models of connected homotopy 2-types [25]. Explicitly, we have the equivalence

$$\mathbf{H_2Typ}^c \simeq Ho(\mathbf{xm}).$$

In other words, we can work with crossed modules which are computationally favorable in comparison to working directly with connected homotopy 2-types.

One difficulty that arises in studying $Ho(\mathbf{xm})$ is when working with the maps of this category. To study such a map, Quillen's homotopy theory requires a replacement of the domain of the map by a type of free crossed module which can be difficult to compute. For crossed modules H, G, Ettore Aldrovandi and Behrang Noohi constructed objects called *butterflies* from H to G which naturally form a groupoid denoted by $\mathsf{B}(\mathsf{H}, \mathsf{G})$ (see [1] and [24]). The nomenclature is due to the shape of the objects. The following theorem shows that the connected components of the groupoid $\mathsf{B}(\mathsf{H}, \mathsf{G})$ model morphisms of the homotopy category of crossed modules.

Theorem. Let H and G be crossed modules. Then we have the bijection

$$[\mathsf{H}, \mathsf{G}]_{\mathbf{xm}} \cong \pi_0 \mathsf{B}(\mathsf{H}, \mathsf{G})$$

Following Behrang Noohi's work, we will formulate butterflies in Chapter 3 by unfolding a *pushout* of crossed modules. By construction, butterflies are completely algebraic objects which do not require a cofibrant replacement. The purpose of this dissertation is to begin to formulate a similar construction of pushouts and butterflies for a category $\mathbf{Xc}$ of objects called crossed complexes which generalize crossed modules (see Chapter 4).

There have been relatively recent developments which have extended the model of homotopy 2-types to certain homotopy n-types using crossed complexes (see [3, 8]). Crossed complexes carry a model structure which generalizes the Moerdijk-Svensson model structure on crossed modules. Moreover, to work with the morphisms of the homotopy category of crossed complexes, the domain must be replaced by a type of free crossed complex which is difficult to compute. The goal of the following work has been to adopt the butterfly model of crossed modules to crossed complexes. The current progress has been to adopt pushouts and butterflies to the full subcategory of crossed complexes composed of reduced n-crossed complexes. For reduced n-crossed complexes H, G, we have constructed objects called *n-butterflies* from H to G. These objects naturally form a groupoid denoted by $\mathsf{B}^n(\mathsf{H}, \mathsf{G})$. Our main result is the following theorem which is stated at the end of Section 5.5.

Theorem. Let H and G be reduced n-crossed complexes. Then we have the bijection

$$[\mathsf{H}, \mathsf{G}]_{\mathbf{Xc}} \cong \pi_0 \mathsf{B}^n(\mathsf{H}, \mathsf{G})$$

Thus, n-butteflies model morphisms between reduced n-crossed complexes in the homotopy category of crossed complexes. Similarly to Behrang Noohi's work, we will formulate n-butterflies in Chapter 5 by unfolding a *n-pushout* of crossed complexes below a particular morphism. The rest of the chapter is devoted to exploring results which will ultimately lead to the main result above.

CHAPTER 2

2-GROUPS AND 2-TYPES

The fact that connected homotopy 2-types are modeled by 2-groups is well-known and an even older fact is that connected homotopy 1-types are modeled by groups. Unfortunately, 2-groups are a categorical model in comparison to groups which are purely algebraic. In the 1940s, J.H.C Whitehead introduced crossed modules which are objects that extended the relationship of 2-groups to a purely algebraic model. The key component in producing this relationship is by introducing a model structure on the category of crossed modules. Our intent in this chapter is to give an overview of both 2-groups and crossed modules, and to formulate how they model homotopy 2-types. Before we formally introduce 2-groups, we will define the more general notion of 2-groupoids.

2.1 2-Groupoids

In set theory, there is a formal way to construct a set where an equivalence relation becomes equality by the method of quotients; however, this method deletes important information. A category[1] is a nice setting to be able to analyze notions of equivalence without losing pertinent information. In the category of topological spaces, we can view two objects as equivalent if they are isomorphic which translates to being homeomorphic in the topological language. From classical algebraic topology, we know that even this notion of equality can be to rigid and we might want to relax the definition of equivalence. For example, we may only want to consider isomorphism up to homotopy. Explicitly, X and Y are *weakly equivalent* if they are homotopy equivalent. Since the morphisms between objects of a category form sets and homotopy is an equivalence relation which respects composition, we can quotient the sets of morphisms by homotopy. The resulting category is a quotient category in which weak equivalences are actually isomorphisms. In this particular case, the quotient category is called the homotopy category.

Although the homotopy category is a fruitful endeavor, the process of taking the quotient deletes important information that could be necessary for a more extensive classification. The idea of higher categories is precisely to retain this information. We begin here with 2-categories.

[1] For a review of the basic notions of category theory which will be used throughout the thesis (see [22] or [4]).

Definition 2.1.1. A 2-**category** $\mathcal{C}$ is data consisting of a class $\mathcal{C}_0$ of *objects*; for each ordered pair of objects (A, B), a small category $\mathcal{C}(A, B)$ whose objects are 1-*morphisms*, denoted $f : A \to B$, and morphisms are 2-*morphisms*, denoted by $\alpha : f \Rightarrow g$; for each triple $A, B, C \in \mathcal{C}_0$, a bifunctor

$$c_{ABC} : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C);$$

for each $A \in \mathcal{C}_0$, a functor

$$u_A : * \to \mathcal{C}(A, A)$$

where $*$ is the category with one object $*$ and one arrow 1_*. The data satisfies the following conditions.

Associativity: For any $A, B, C, D \in \mathcal{C}_0$,

$$c_{ABD} \circ (1 \times c_{BCD}) = c_{ACD} \circ (c_{ABC} \circ 1) : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \times \mathcal{C}(C, D) \to \mathcal{C}(A, D).$$

Units: For any $A, B \in \mathcal{C}_0$,

$$1_{\mathcal{C}(A,B)} \cong c_{AAB} \circ \left(u_A \times 1_{\mathcal{C}(A,B)}\right) : \mathcal{C}(A, B) \cong * \times \mathcal{C}(A, B) \to \mathcal{C}(A, B)$$

and

$$1_{\mathcal{C}(A,B)} \cong c_{ABB} \circ \left(1_{\mathcal{C}(A,B)} \times u_B\right) : \mathcal{C}(A, B) \cong \mathcal{C}(A, B) \times * \to \mathcal{C}(A, B).$$

We define $1_A = u_A(*)$ and $1_{1_A} = u_A(1_*)$. The composition on 1-morphisms imposed by the bifunctor will be denoted by the usual $\circ$. With the axioms of a 2-category, the objects and 1-morphisms of a 2-category $\mathcal{C}$ form a category which we will denote by $\mathcal{C}_1$.

Since $\mathcal{C}(A, B)$ is a category, there is a composition of 2-morphisms called vertical composition which will be denoted by $\Diamond$. Diagrammatically,

$$\begin{tikzcd} A \arrow[r, bend left=70, "f"] \arrow[r, "g"'] \arrow[r, bend right=70, "h"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & B \end{tikzcd} = \begin{tikzcd} A \arrow[rr, bend left, "f"] \arrow[rr, bend right, "h"'] \arrow[rr, phantom, "\Downarrow\beta\Diamond\alpha"] & & B \end{tikzcd}$$

The bifunctor c_{ABC} induces another composition on 2-morhisms called horizontal composition which is denoted by $\circ$. Diagrammatically,

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "q"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd} = \begin{tikzcd} A \arrow[rr, bend left, "g\circ f"] \arrow[rr, bend right, "q\circ p"'] \arrow[rr, phantom, "\Downarrow\beta\circ\alpha"] & & C \end{tikzcd}$$

Compatibility of the two compositions of 2-morphisms is given by the following theorem.

Theorem 2.1.2. A 2-category $\mathcal{C}$ satisfies the *Interchange Law*:

Let

$$\begin{tikzcd}[column sep=large]
A \arrow[r, bend left=60, "f"] \arrow[r, "g"'] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\gamma"] & B \arrow[r, bend left=60, "p"] \arrow[r, "q"'] \arrow[r, bend right=60, "r"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\beta"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\delta"] & C
\end{tikzcd}$$

be a diagram in $\mathcal{C}$. Then

$$(\delta \circ \gamma) \diamondsuit (\beta \circ \alpha) = (\delta \diamondsuit \beta) \circ (\gamma \diamondsuit \alpha)$$

Proof. Functoriality of c. □

As usual, there is proper way to compare a 2-category $\mathcal{C}$ to another 2-category $\mathcal{D}$.

Definition 2.1.3. Let $\mathcal{C}, \mathcal{D}$ be 2-categories. A **2-functor** F is data consisting of a well-defined assignment FA for each $A \in \mathcal{C}$; for any $A, B \in \mathcal{C}$, a functor

$$F_{AB} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$$

which satisfies the following properties.

Functoriality of Composition: For $A, B, C \in \mathcal{C}_0$,

$$F_{AB} \circ c_{ABC} = c_{FAFBFC} \circ (F_{AB} \times F_{BC}) \,.$$

Functoriality of Units: For $A \in \mathcal{C}_0$,

$$F_{AA} \circ u_A = u_{FA}.$$

The category of small 2-categories along with 2-functors as morphisms is denoted by **2Cat**. Similarly to natural transformations between functors of categories, there are 2-*natural transformations* between 2-functors. Also, due to the additional data of 2-morphisms there is a notion of morphisms between 2-natural transformations called *modifications*. For our needs, we do not need to formally define either of these notions, but only realize them as morphisms between the respective objectives. The formal definitions can be found in either of the sources [4] or [22].

Now we list a few important 2-categories beginning with the archetypical example.

Example 2.1.4. The class of small categories with functors as 1-morphisms, and natural transformations as 2-morphisms form a 2-category. Likewise, small 2-categories as objects, 2-functors as 1-morphisms, and 2-natural transformations as 2-morphisms form a 2-category.

Remark 2.1.5. One might suggest at this point that the data of small 2-categories, 2-functors,2-natural transformations, and 2-modifications should form a 3-*category*. Indeed, this is the archetypical example for defining 3-categories. In fact, following this approach one can recursively define n-*categories*; however, we will not define them in this manner. Instead, we will define them using the approach of globular sets in section 4.1.

Example 2.1.6. The class of topological spaces with continuous maps as 1-morphisms and homotopy classes of homotopies as 2-morphisms form a 2-category. The 2-morphisms must be homotopy classes in order for the associative axiom to be satisfied.

Example 2.1.7. Let $f : H \to G$ be a group homomorphism. Then f defines an equivalence relation on the elements of G. Explicitly, $a, b \in \mathsf{G}$ are equivalent if and only if $f(a) = f(b)$. We can form a 2-category $\mathcal{C}$ by taking $*$ to be the only object, G to be the set of 1-morphisms $\mathcal{C}(*, *)$, and the equivalence relation on G to define the 2-morphisms. Clearly, every object has an identity map and composition of 1-morphisms is identified with the algebraic structure on G. Vertical composition is defined by transitivity of the equivalence relation and horizontal composition is defined since f is a homomorphism. Notice that every 1-morphism and 2-morphism has an inverse since G is a group and equivalence relations are symmetric, respectively.

2-categories with the additional requirement that every 1-morphism and 2-morphism has an inverse are common and will be of particular interest throughout our discussion.

Definition 2.1.8. A **2-groupoid** is a 2-category in which both 1-morphisms and 2-morphisms are invertible. A **2-group** is a 2-groupoid with only one object. The category with small 2-groupoids (or small 2-groups) along with 2-functors as morphisms will be denoted by **2Grpd** (or **2Grp**).

A beneficial example which we describe below begins an approach to rigidifying topological spaces by first forming a categorical representative. Conversely, this example gives intuition on how we should be defining higher categorical structures and their possible variations.

Example 2.1.9. Let X be a topological space. Recall that a path from a point a to a point b of X is a continuous function $f : I \to X$ where I is the unit interval and $f(0) = a, f(1) = b$. We will abuse notation and denote the path f by $f : a \to b$. Every object a has an identity path $1_a : a \to a$ defined by $1_a(t) = a$ for all $t \in I$. Composition of two paths $f : a \to b, g : b \to c$ is given by the path $g \circ f : I \to X$ defined by

$$g \circ f(t) = \begin{cases} f(2t) : 0 \leq t \leq 1/2 \\ g(2t-1) : 1/2 < t \leq 1 \end{cases}$$

Unfortunately, composition is not associative. As a counterexample, suppose f, g are defined as before where the points b and c do not equal and consider a path $h : c \to d$ in X. Then

$$((h \circ g) \circ f)(1/2) = b \neq c = (h \circ (g \circ f))(1/2)$$

However, there is a homotopy between $\alpha : I \times I \to X$ from $(h \circ g) \circ f$ to $h \circ (g \circ f)$ which is relative to the ordered pair of points (a, d). Explicitly,

$$H(s,t) = \begin{cases} (1-s)f(2t) + sf(4t) & : 0 \leq t \leq \frac{2-s}{4} \\ (1-s)g(4t-2) + sg(4t-1) & : \frac{2-s}{4} \leq t \leq \frac{3-s}{4} \\ (1-s)h(4t-3) + sh(2t-1) & : \frac{3-s}{4} \leq t \leq 1 \end{cases}$$

So defining objects to be points of X and 1-morphisms from a to b to be elements of the set $\pi_1(X, (a, b))$ of homotopy classes of paths relative to (a, b), the associativity condition of a category is satisfied and we have defined a category $\Pi_1(X)$. In fact, since paths are invertible, this is a groupoid called the *fundamental groupoid*. Notice that for a singleton a, $\Pi_1(X)(a, a)$ is the fundamental group $\pi_1(X, a)$ of X. We can also include the information of homotopies between the homotopy classes of paths by storing the additional data as 2-morphisms. Horizontal and vertical composition is clear, but for associativity to be satisfied, we must again take 2-morphisms to be the homotopy classes of homotopies. The data of objects, 1-morphisms, and 2-morphisms form a 2-category. Since paths and homotopies are invertible, this 2-category is in fact a 2-groupoid called the *fundamental* 2*-groupoid* of the space X and denoted $\Pi_2(X)$.

In order for the above example to satisfy the associativity and unital axioms of a 2-category, we had to take the homotopy classes of the 1-morphisms and 2-morphisms. Taking the quotient of the 2-morphisms was inevitable since we were already choosing to ignore higher homotopies; however, taking the quotient of 1-morphisms seems excessive. In fact, there are plenty of examples of 2-categories where the associativity and unital axioms of 1-morphisms are only satisfied up to 2-isomorphism. Enough to hint that 2-categories and 2-groupoids may be somewhat strict for certain cases and that we should have a notion of *weak* 2*-categories* and *weak* 2*-groupoids*.

Informally, a weak 2-category, also commonly referred to as a bicategory, is the data of a 2-category with the amendment that the associativity and unital axioms of 1-morphisms hold up to 2-isomorphisms. Then analogs of 2-functors, 2-transformations, and 2-modifications can be respectively formulated as bifunctors, bitransformations and bimodifications to preserve the weakened data. In particular, the formulation of the analog of a 2-equivalence allows for the following theorem which may settle some nerves.

Coherence Theorem 2.1.10. [16] Every bicategory is biequivalent to a 2-category.

Due to the Coherence Theorem and our long-term goals, we will only be considering the stricter definition; in particular, we are mainly interested in 2-groupoids. We now abstract the notation from example 2.1.9 and have the following terminology.

Definition 2.1.11. Let $\mathcal{G}$ be a small 2-groupoid. The **connected components** of $\mathcal{G}$ is the set $\pi_0(\mathcal{G})$ of 1-isomorphism classes of the objects of $\mathcal{G}$. The **fundamental groupoid** of $\mathcal{G}$ is the groupoid $\Pi_1(\mathcal{G})$ with the same objects as $\mathcal{G}$ and the morphisms are 2-isomorphism classes of the 1-morphisms of $\mathcal{G}$. For an object $A \in \mathcal{G}$, the group of automorphisms is denoted by $\pi_1(\mathcal{G}, A)$ and the group of automorphisms of the identity 1_A is denoted by $\pi_2(\mathcal{G}, A)$.

Analogous to the model structure on the category **Top** (see Theorem A.1.27), we define the following morphisms of 2-groupoids.

Definition 2.1.12. Let $\mathcal{H}, \mathcal{G}$ be 2-groupoids. A 2-functor $F : \mathcal{H} \to \mathcal{G}$ is a **weak equivalence** if F induces a bijection $\pi_0(\mathcal{H}) \to \pi_0(\mathcal{G})$ of sets and isomorphisms $\pi_i(\mathcal{H}, A) \to \pi_i(\mathcal{G}, F(A))$ of groups for $i = 1, 2$ and every $A \in \mathcal{H}_0$.

Definition 2.1.13. Let $\mathcal{H}, \mathcal{G}$ be 2-groupoids. A 2-functor $F : \mathcal{H} \to \mathcal{G}$ is a **fibration** if for any 1-morphism $f : A_1 \to A_2$ in $\mathcal{H}$, $B \in \mathcal{G}_0$ and commutative diagram

$$\begin{tikzcd}
 & F(A_1) \arrow[d, "F(f)"] \\
B \arrow[ur, "g"] \arrow[r, "h"'] \arrow[ur, phantom, shift right=3ex, "\alpha\,\Downarrow"] & F(A_2)
\end{tikzcd}$$

in $\mathcal{G}$ there exists a 2-morphism $\tilde{\alpha} : f \circ \tilde{g} \Rightarrow \tilde{h}$ in $\mathcal{H}$ such that $F(\tilde{\alpha}) = \alpha, F(\tilde{g}) = g$, and $F(\tilde{h}) = h$.

Taking the cofibrations to be the morphisms of 2-groupoids which have the left lifting property with respect to trivial fibrations (see Theorem A.1.13), we have the following theorem.

Theorem 2.1.14. [23] There is a model structure on the category of 2-groupoids where weak equivalences, fibrations and cofibrations are as defined above.

We will refer to this model structure on the category of 2-groupoids as the Moerdijk-Svensson model structure. Recall that the category of topological spaces exhibits a model structure in which weak equivalences are the weak homotopy equivalences. This model structure is commonly referred to as the Quillen model structure and we will denote the model category by $\mathbf{Top}_Q$.

Definition 2.1.15. The objects of $Ho(\mathbf{Top}_Q)$ which have trivial n-homotopy groups for $n > 2$ are called **homotopy 2-types**. The full subcategory of $Ho(\mathbf{Top}_Q)$ consisting of homotopy 2-types (or connected homotopy 2-types) is denoted by $\mathbf{H}_2\mathbf{Typ}$ (or $\mathbf{H}_2\mathbf{Typ}^c$).

We next state the main theorem of the section. This theorem relies on defining a nerve functor N from **2Grpd** to **sSet** which is then followed by the geometric realization $|\cdot|$ to land in the category of topological spaces. Fortunately, these functors respect the Quillen model structure so, to be brief, we state the interesting result.

Theorem 2.1.16. [25] There is a fully faithful functor

$$|N(\cdot)| : Ho(\mathbf{2Grpd}) \to Ho(\mathbf{Top}_Q)$$

which induces the equivalence

$$Ho(\mathbf{2Grpd}) \simeq \mathbf{H}_2\mathbf{Typ}$$

and restricts to the equivalence

$$Ho(\mathbf{2Grp}) \simeq \mathbf{H}_2\mathbf{Typ}^c_*.$$

where $\mathbf{H}_2\mathbf{Typ}^c_*$ is the category of *pointed* connected 2-types.

2.2 Crossed Modules

As 2-groupoids are much more algebraic objects than spaces, the equivalence in theorem 2.1.16 is a step in the right direction for rigidifying spaces algebraically. For homotopy 2-types, there are actually purely algebraic models called crossed modules which were introduced by J.H.C. Whitehead in [30] and [31]. These will be the main objects throughout the rest of the chapter and will continue to be a cornerstone in our discussion.

Definition 2.2.1. A **crossed module** $[\mathsf{C} : \partial]$ is a group homomorphism $\partial : \mathsf{C}_2 \to \mathsf{C}_1$ with a right action of C_1 on C_2, denoted by a^x, which descends to conjugation under ∂ (CM1) and lifts to conjugation when restricted to $\partial(\mathsf{C}_2)$ (CM2).

$$\text{CM1:} \quad \partial(a^x) = x^{-1}\partial(a)x \text{ for all } x \in \mathsf{C}_1, a \in \mathsf{C}_2$$

$$\text{CM2:} \quad a^{\partial(b)} = b^{-1}ab \text{ for all } a, b \in \mathsf{C}_2$$

A morphism $f : [\mathsf{C} : \partial] \to [\mathsf{D} : \delta]$ is a commutative diagram

$$\begin{tikzcd} \mathsf{C}_2 \arrow[r, "f_2"] \arrow[d, "\partial"'] & \mathsf{D}_2 \arrow[d, "\delta"] \\ \mathsf{C}_1 \arrow[r, "f_1"'] & \mathsf{D}_1 \end{tikzcd}$$

in which f_2 is f_1-equivariant i.e. $f_2(a^x) = f_2(a)^{f_1(x)}$.

Crossed modules along with the morphisms form a category which we denote by **xm**.

Remark 2.2.2. The descending and lifting properties of the action can be described by the commutativity of the diagrams

CM1:
$$\begin{tikzcd} \mathsf{C}_2 \times \mathsf{C}_1 \arrow[r, "\sigma"] \arrow[d, "\partial \times id_{\mathsf{C}_1}"'] & \mathsf{C}_2 \arrow[d, "\partial"] \\ \mathsf{C}_1 \times \mathsf{C}_1 \arrow[r, "c"'] & \mathsf{C}_1 \end{tikzcd}$$

CM2:
$$\begin{tikzcd} \mathsf{C}_2 \times \mathsf{C}_2 \arrow[r, "id_{\mathsf{C}_2} \times \partial"] \arrow[dr, "c"'] & \mathsf{C}_2 \times \mathsf{C}_1 \arrow[d, "\sigma"] \\ & \mathsf{C}_2 \end{tikzcd}$$

where σ is the action and c is the conjugation action by the right.

The property CM1 is the fact that ∂ is C_1-equivariant and property CM2 is commonly referred to as the *Peiffer Identity*. For convenience, we will usually drop the homomorphism from the notation of a crossed module.

Example 2.2.3. Let G be a group and $c : G \to Aut(G)$ be the group homomorphism which sends an element $g \in G$ to the inner automorphism c_g. Explicitly, $c_g(a) = g^{-1}ag$ for all $a \in G$. Given an automorphism α of G, the right action on G is defined by $g^\alpha = \alpha(g)$. For all $a \in G$, we have that

$$\begin{aligned} c(g^\alpha)(a) &= c(\alpha(g))(a) \\ &= \alpha(g)^{-1} a \alpha(g) \\ &= \alpha(g^{-1})\alpha(\alpha^{-1}(a))\alpha(g) \\ &= \alpha(g^{-1}\alpha^{-1}(a)g) \\ &= \alpha(c_g(\alpha^{-1}(a))) \\ &= (\alpha \circ c_g \circ \alpha^{-1})(a) \end{aligned}$$

which gives CM1 and CM2 follows from the equalities $g^{c(h)} = c_h(g) = h^{-1}gh$ where $h \in G$.

The second example of a crossed module that we present was one of J.H.C Whitehead's driving examples for defining crossed modules.

Example 2.2.4. Let (X, A) be a pair of spaces with $a \in A \subset X$. Recall that $\pi_2(X, A, a)$ is the group of homotopy classes of continuous maps $\alpha : (I^2, \partial I^2, \overline{\partial I^2 - I}) \to (X, A, a)$ where I is considered as the face of ∂I^2 with second coordinate 0. Define a map $\partial : \pi_2(X, A, a) \to \pi_1(A, a)$ which sends an element α of $\pi_2(X, A, a)$ to its boundary $\alpha|_I : I \to A$. Let J^2 be the sub-square of I^2 whose face J with second coordinate 0 is centered in I. The action of a loop $f \in \pi_1(A, a)$ on $\alpha \in \pi_2(X, A, a)$ is the homotopy class of a continuous map $\alpha^f : (I^2, \partial I^2, \overline{\partial I^2 - I}) \to (X, A, a)$ which restricts to f on the sub-square J^2 and the boundary $\partial(\alpha) = \alpha(I)$ is the loop $f^{-1} \circ \alpha(J) \circ f$.

The above example will be important later as it can be generalized to a filtration (see Example 4.2.8) of a space rather than just a pair of spaces. This gives the archetypical example of a crossed complex.

Example 2.2.5. Given a 2-group $\mathcal{G}$ with the single object $*$, we can define a crossed module $\partial : G_2 \to G_1$ where G_1 is the group of 1-morphisms and G_2 is the group $\cup_{f \in \mathcal{G}(*,*)} \mathcal{G}(1_*, f)$ with horizontal composition as the operation. The map ∂ sends a morphism $\alpha : 1_* \Rightarrow f$ to its target f which is a group homomorphism since horizontal composition of 2-morphisms is compatible with composition of 1-morphisms.

Using the construction in the above example, 2-groups are completely characterized by crossed modules up to isomorphism and vice versa. Moreover, the respective morphisms are characterized as well. Formally, we have the following well-known fact.

Theorem 2.2.6. The category of crossed modules is equivalent to the category of 2-groups.

In light of this theorem, we will usually denote crossed modules by the familiar notation G, H used in group theory.

2.3 Homotopy Enrichment

We now introduce the notion of homotopy between two morphisms of crossed modules. Note the similarity to a chain homotopy of chain complexes focused in degrees one and two.

Definition 2.3.1. Let $f, g : [\mathsf{H} : \partial] \to [\mathsf{G}, \delta]$ be morphisms of crossed modules. A **homotopy** from f to g is a crossed homomorphism $\phi : H_1 \to G_2$ (i.e. $\phi(xy) = \phi(x)^{f_1(y)}\phi(y)$) such that

$$f_1(x)g_1(x)^{-1} = \delta(\phi(x)) \quad \text{for all} \quad x \in \mathsf{H}_1$$

$$f_2(a)g_2(a)^{-1} = \phi(\partial(a)) \quad \text{for all} \quad a \in \mathsf{H}_2$$

Considering the additional data of homotopies enriches the category of crossed modules. More explicitly, we have the following fact.

Theorem 2.3.2. Crossed modules along with morphisms as 1-morphisms and homotopies as 2-morphisms form a 2-category.

In fact, the equivalence of theorem 2.2.6 can be extend to a 2-equivalence when the category of 2-groups is considered as a 2-category. A complete exposition of both the above and below theorems can be found in [26].

Theorem 2.3.3. The 2-category of crossed modules is 2-equivalent to the 2-category of 2-groups.

Although we will not go through all the details, we will mention that vertical composition of homotopies is defined by multiplication of the homotopy maps in the codomain. All homotopies are invertible since the codomains of the homotopy maps are groups. Thus, the categories of 1-morphisms and 2-morphisms are groupoids.

When it is necessary to make the distinction between the category and the 2-category of crossed modules, we will denote the 2-category of crossed modules by $\underline{\mathbf{xm}}$ to make explicit that the category is enriched in groupoids. Since these groupoids will be of particular interest as we shall shortly explain, we will follow Behrang Noohi's terminology below.

Definition 2.3.4. Let $\mathsf{H}, \mathsf{G} \in \mathbf{xm}_0$. The **mapping groupoid** from H to G is the groupoid $\underline{\mathbf{xm}}(\mathsf{H}, \mathsf{G})$ consisting of morphisms of crossed modules from H to G as objects and homotopies as morphisms.

Theorem 2.3.3 along with section 2.1 begins to give us the impression that crossed modules could be a completely algebraic model of connected homotopy 2-types. To make this precise, we need to have a coherent homotopy theory.

Definition 2.3.5. Let $[\mathsf{G} : \partial]$ be a crossed module. The **fundamental homotopy group** of $[\mathsf{G} : \partial]$ is the group $\pi_1(\mathsf{G}) = \text{coker}\partial$ and the **2-homotopy group** is the group $\pi_2(\mathsf{G}) = \ker \partial$.

Definition 2.3.6. Let $f : \mathsf{H} \to \mathsf{G}$ be a morphism of crossed modules. Then f is a **weak equivalence** if the induced maps of homotopy groups are isomorphisms and a **fibration** if f_0, f_1 are surjective maps.

As usual, the *cofibrations* are determined by the left lifting property with respect to trivial fibrations.

Theorem 2.3.7. [23] There exists a model structure on **xm** with weak equivalences, fibrations and cofibrations as defined above.

The crossed module $1 : \mathbf{1} \to \mathbf{1}$ of trivial groups is the terminal (initial and final) object of **xm**. Clearly, every object is fibrant; however, not every object is cofibrant.

We now state the main theorem of the section.

Theorem 2.3.8. [24] The equivalence between **xm** and **2Grp** preserves the homotopy groups. Explicitly, there is an equivalence of categories

$$Ho(\mathbf{xm}) \simeq Ho(\mathbf{2Grp}).$$

Since the categories of crossed modules and 2-groups are equivalent and, furthermore, the homotopy categories agree with respect to the Moerdijk-Svensson model structure on **2Grp**, we will also refer to the model structure on the category of crossed modules as the Moerdijk-Svensson model structure. Using theorem 2.1.16, we deduce the following corollary.

Corollary 2.3.9. There is an equivalence of categories

$$Ho(\mathbf{xm}) \simeq \mathbf{H}_2\mathbf{Typ}_*^c.$$

Remark 2.3.10. The above corollary shows that to study connected homotopy 2-types we can conveniently work in the model category **xm**. So the important morphisms to study between crossed modules are the morphisms of the homotopy category in comparison to the stricter morphisms of **xm**.

In order to understand the morphisms of the homotopy category, we first study them while ignoring the quotient with respect to homotopy.

Definition 2.3.11. Let H and G be crossed modules. A **weak morphism** from H to G is a morphism in $\mathbf{xm}(\mathsf{Q}, \mathsf{G})$ where Q is a cofibrant replacement of H.

In order to understand the weak morphisms, we need to have good descriptions of fibrant and cofibrant objects. Fortunately, every object in **xm** is fibrant; however, describing cofibrant objects will take some work.

Theorem 2.3.12. [24] A morphism of crossed modules $f : \mathsf{H} \to \mathsf{G}$ is a trivial fibration if and only if f_1 is surjective and $\mathsf{H}_2 \cong \mathsf{H}_1 \times_{\mathsf{G}_1} \mathsf{G}_2$.

Using this description of trivial fibrations, Behrang Noohi was able to completely characterize the cofibrant objects.

Theorem 2.3.13. [24] A crossed module given by $\partial : \mathsf{G}_2 \to \mathsf{G}_1$ is cofibrant in the Moerdijk-Svensson model structure if and only if G_1 is free.

So not every object is cofibrant and computing cofibrant objects requires a calculation of a free group.

CHAPTER 3

BUTTERFLIES

The key to forming the relationship between connected homotopy 2-types and crossed modules in the prior chapter was introducing a model structure on crossed modules. Specifically, the category of connected homotopy 2-types is modeled by the homotopy category of crossed modules with respect to the defined model structure. Although we now have an algebraic model of connected homotopy 2-types, there are some complications. In particular, to study the morphisms of this homotopy category, we really need to study weak morphisms. These require computing a cofibrant replacement which we would rather avoid. The intent of this chapter is to introduce butterflies which are purely algebraic models of the weak morphisms. In other words, butterflies are nice models which avoid computing a cofibrant replacement. We will begin with defining precisely what morphisms we would like to model. These morphisms form a groupoid which is called the *derived mapping groupoid*.

3.1 Derived Mapping Groupoid

The result of theorem 2.3.8 gives us a description of the proper morphisms to study when modeling connected homotopy 2-types and, furthermore, 2-groups with the Moerdijk-Svensson model structure. Specifically, we want to find an algebraic model of the weak morphisms between 2-groups. First, lets decompose the weak morphisms.

Specifically, for 2-groups $\mathcal{H}, \mathcal{G}$ and their respective images H, G under the equivalence, we have the bijection

$$[\mathcal{H}, \mathcal{G}]_{\mathbf{2Grp}} \cong [\mathsf{H}, \mathsf{G}]_{\mathbf{xm}}$$

In particular,

$$[\mathsf{H}, \mathsf{G}]_{\mathbf{xm}} = \mathbf{xm}(QR\mathsf{H}, QR\mathsf{G})/\simeq$$

where Q, R are the cofibrant and fibrant replacement functors, respectively. In fact, theorem A.3.9 allows us to simplify the right side of the above isomorphism to

$$\begin{aligned}[\mathsf{H}, \mathsf{G}]_{\mathbf{xm}} &= \mathbf{xm}(QR\mathsf{H}, QR\mathsf{G})/\simeq \\ &\cong \mathbf{xm}(Q\mathsf{H}, R\mathsf{G})/\simeq \\ &\cong \mathbf{xm}(Q\mathsf{H}, \mathsf{G})/\simeq\end{aligned}$$

where the second isomorphism follows since all crossed complexes are fibrant.

So understanding the morphisms of the homotopy category of 2-groups (or equivalently, crossed modules) amounts to understanding the *derived* set of morphisms

$$\underline{\mathbf{Rhom}}(\mathsf{H},\mathsf{G}) = \mathbf{xm}(\mathsf{Q},\mathsf{G})$$

where Q is a cofibrant replacement of H. These are what we call weak morphisms.

Following Behrang Noohi's work in [24], we have the following definition.

Definition 3.1.1. Let H and G be crossed modules and fix Q to be a cofibrant replacement of H. The **derived mapping groupoid** of maps from H to G, denoted by $\underline{\mathbf{Rhom}}(\mathsf{H},\mathsf{G})$, is the groupoid $\underline{\mathbf{xm}}(\mathsf{Q},\mathsf{G})$.

Objects of the derived mapping groupoid are just fractions of the form

$$\begin{tikzcd} & \mathsf{Q} \arrow[dl, two heads, "p"', "\simeq"] \arrow[dr, "f"] & \\ \mathsf{H} \arrow[rr, dotted, "w"'] & & \mathsf{G} \end{tikzcd} \tag{3.1}$$

where $\mathsf{Q} \xrightarrow[p]{\simeq} \mathsf{H}$ is a cofibrant replacement of H.

Theorem 3.1.2. [24] Let H and G be crossed modules. The there is a bijection

$$[\mathsf{H},\mathsf{G}]_{\mathbf{xm}} \cong \pi_0 \underline{\mathbf{Rhom}}(\mathsf{H},\mathsf{G}).$$

We have reduced studying weak morphisms to computing cofibrant replacements. In theorem 2.3.13, Behrang Noohi realized that this relies on computing a free object which we would prefer to avoid. As cofibrant objects are not very manageable, we would like to construct a crossed module E algebraically which corresponds to the weak morphism ω i.e. we have a new fraction

$$\begin{tikzcd} & \mathsf{Q} \arrow[d] \arrow[ddl, two heads, "p"', "\simeq"] \arrow[ddr, "f"] & \\ & \mathsf{E} \arrow[dl, two heads, "\simeq"', "p'"] \arrow[dr, "f'"'] & \\ \mathsf{H} \arrow[rr, dotted, "w"'] & & \mathsf{G} \end{tikzcd} \tag{3.2}$$

Ideally, E should be easier to compute in comparison to Q. Thus, giving a manageable model of the weak morphisms of crossed modules.

Theorem 3.1.3. Let $[\mathsf{H} : \partial]$ and $[\mathsf{G} : \delta]$ be crossed modules. Then the group homomorphism

$$\mathsf{H}_2 \times \mathsf{G}_2 \xrightarrow{\partial \times \delta} \mathsf{H}_1 \times \mathsf{G}_1$$

is a crossed module $[\mathsf{H} \times \mathsf{G} : \partial \times \delta]$, called the **product crossed module**, with the induced action.

Proof. Clear. $\square$

Definition 3.1.4. A crossed module G is **acyclic** if all homotopy groups are trivial.

Proposition 3.1.5. Let $[\mathsf{H} : \partial]$ and $[\mathsf{G} : \delta]$ be crossed modules. If G is acyclic, then the projection of crossed modules $\pi_1 : \mathsf{H} \times \mathsf{G} \to \mathsf{H}$ is a trivial fibration. Similarly, if H is acyclic, then the projection of crossed modules $\pi_2 : \mathsf{H} \times \mathsf{G} \to \mathsf{G}$ is a trivial fibration.

Proof. Since the homomorphism of the crossed module $\mathsf{H} \times \mathsf{G}$ is the product map, we have that $\pi_1(\mathsf{H} \times \mathsf{G}) = \operatorname{coker}\partial \times \operatorname{coker}\delta = \pi_1(\mathsf{H}) \times \pi_1(\mathsf{G})$ and $\pi_2(\mathsf{H} \times \mathsf{G}) = \ker \partial \times \ker \delta = \pi_2(\mathsf{H}) \times \pi_2(\mathsf{G})$. Since G is acyclic, the right side of these products are zero. The fact that the projection π_1 is a fibration is clear. A similar argument can be applied for when H is acyclic. $\square$

Remark 3.1.6. From the proposition above, one should think of the product crossed module $\mathsf{H} \times \mathsf{G}$ as 'smack' in the middle of H and G, at least up to weak equivalence.

From diagram 3.1, we inherit a morphism of crossed modules $p \times f : \mathsf{Q} \to \mathsf{H} \times \mathsf{G}$ which we will denote by ∇^f since p is fixed. Moreover, there is an induced factorization

$$\begin{tikzcd}
 & \mathsf{Q} \arrow[d, dashed, "\nabla^f"] \arrow[ddl, "p"', "\simeq"] \arrow[ddr, "f"] & \\
 & \mathsf{H} \times \mathsf{G} \arrow[dl, dashed, "\pi_1"] \arrow[dr, dashed, "\pi_2"'] & \\
\mathsf{H} \arrow[rr, dotted, "w"'] & & \mathsf{G}
\end{tikzcd}$$

By the two out of three property of weak equivalences, if either ∇^f or π_1 is a weak equivalence, the other is guaranteed to be a weak equivalence. Unfortunately, neither is necessarily a weak equivalence; otherwise, $\mathsf{H} \times \mathsf{G}$ would be a suspect for E. So we want something "closer" to Q. Notice that such a crossed complex E would need to factor ∇^f by the universality of the product crossed module.

3.2 Pushouts of Crossed Modules

In order to find the correct crossed module E to complete the diagram 3.2 we would like to form a crossed module by "pushing out" the morphism ∇^f. We will first introduce this operation generally which was addressed my Behrang Noohi in [24]. We will rug through a lot of the details since our later work will benefit from them.

Given a morphism of crossed modules $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$, we would like to find a pushout of the solid diagram

$$\begin{tikzcd} \mathsf{H}_2 \arrow[r, "f_2"] \arrow[d, "\partial"'] & \mathsf{G}_2 \arrow[d, dashed] \\ \mathsf{H}_1 \arrow[r, dashed] & \mathsf{E} \end{tikzcd} \tag{3.3}$$

In particular, we would like to find a crossed module $[\mathsf{G}_2 \to \mathsf{E} : \partial^f]$ which satisfies the commutative diagram 3.3. As we will see, the embedding $[\mathsf{G}_2 \to \mathsf{H}_1 \ltimes \mathsf{G}_2]$ forms a crossed module and there is an embedding $\mathsf{H}_1 \to \mathsf{H}_1 \ltimes \mathsf{G}_2$; however, this does not properly fill in the diagram 3.2.4. We can force the diagram to commute by forcing $(\partial(h), 1)$ to equal $(1, f_2(h))$. In other words, we consider the quotient of $\mathsf{H}_1 \ltimes \mathsf{G}_2$ by $N = \left\{(\partial(h)^{-1}, f_2(h)) | h \in \mathsf{H}_2\right\}$. First, we need to check that N is normal in $\mathsf{H}_1 \ltimes \mathsf{G}_2$.

Following Behrang Noohi's hint, notice that G_2, embedded in $\mathsf{H}_1 \ltimes \mathsf{G}_2$, centralizers N. Explicitly, for $g \in \mathsf{G}_2$ and for $h \in \mathsf{H}_2$, we have

$$\begin{aligned}
(1,g)^{-1}(\partial(h)^{-1}, f_2(h))(1,g) &= (1,g^{-1})(\partial(h)^{-1}, f_2(h))(1,g) \\
&= (\partial(h)^{-1}, (g^{-1})^{\partial(h)^{-1}} f_2(h))(1,g) \\
&= (\partial(h)^{-1}, (g^{-1})^{\partial(h)^{-1}} f_2(h)g) \\
&= (\partial(h)^{-1}, (g^{-1})^{f_2(h)^{-1}} f_2(h)g) \\
&= (\partial(h)^{-1}, (f_2(h)g^{-1}f_2(h)^{-1})f_2(h)g) \\
&= (\partial(h)^{-1}, f_2(h)g^{-1}g) \\
&= (\partial(h)^{-1}, f_2(h))
\end{aligned}$$

Now, for $k \in K$ and $h \in H$,

$$\begin{aligned}
(k,1)^{-1}(\partial(h)^{-1}, f_2(h))(k,1) &= (k^{-1},1)(\partial(h)^{-1}, f_2(h))(k,1) \\
&= (k^{-1}\partial(h)^{-1}, f_2(h))(k,1) \\
&= (k^{-1}\partial(h)^{-1}k, f_2(h)^k)
\end{aligned}$$

$$= ((\partial(h)^{-1})^k, f_2(h)^k)$$
$$= (\partial(h^k)^{-1}, f_2(h)^k).$$

So for $(k, g) \in \mathsf{H}_1 \ltimes \mathsf{G}_2$ and $h \in \mathsf{H}_2$ we have

$$\begin{aligned}(k,g)^{-1}(\partial(h)^{-1}, f_2(h))(k,g) &= ((k,1)(1,g))^{-1}(\partial(h)^{-1}, f_2(h))(k,1)(1,g)\\ &= (1,g)^{-1}(k,1)^{-1}(\partial(h)^{-1}, f_2(h))(k,1)(1,g)\\ &= (1,g)^{-1}(\partial(h^k)^{-1}, f_2(h)^k)(1,g)\end{aligned}$$

where the last equality follows from the fact that G centralizes N. Hence, N is indeed normal in $\mathsf{H}_1 \ltimes \mathsf{G}_2$.

As this "pushout" only requires an equivariant action of H_1 on G_2, we will construct these pushouts generally.

Definition 3.2.1. Given a group K which acts on groups H, G and a K-invariant diagram

$$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "d"'] & G \\ K & \end{tikzcd}$$

where K acts on itself by conjugation, the **generalized semidirect product** of K and G under H is the group $K \ltimes^H G = K \ltimes G/N$ where $N = \{(d(h)^{-1}, p(h)) | h \in H\}$.

We will write elements of $K \ltimes^H G$ in the form $[k, g]$. Composing the usual embeddings of K, G into $K \ltimes G$ with the quotient maps, we have a commutative diagram

$$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "d"'] & G \arrow[d, "d'"] \\ K \arrow[r, "p'"] & K \ltimes^H G \end{tikzcd}$$

Theorem 3.2.2. [24] The morphism $d' : G \to K \ltimes^H G$ is a crossed module. Moreover, if $d : H \to K$ is crossed module, then the morphism $\iota = (p, p')$ is a morphism of crossed modules.

Proof. First, we prove CM1. For $a \in G$ and $[k, g] \in K \ltimes^H G$, we have

$$\begin{aligned}\partial(a)^{[k,g]} &= [k,g]^{-1}\partial(a)[k,g]\\ &= [k^{-1}, (g^{-1})^{k^{-1}}][1,a][k,g]\\ &= [k^{-1}, (g^{-1})^{k^{-1}}][k, a^k g]\end{aligned}$$

$$
\begin{aligned}
&= [1, g^{-1}a^k g] \\
&= [1, a^{[k,g]}] \\
&= \partial(a^{[k,g]})
\end{aligned}
$$

To show CM2 holds, given $a, b \in G$, we have

$$
\begin{aligned}
a^{\partial(b)} &= a^{[1,b]} \\
&= b^{-1}ab \\
&= a^b
\end{aligned}
$$

To show that (p, p') is a morphism of crossed modules, everything is clear except the equivariance. However, for $h \in H$ and $k \in K$, we have

$$
\begin{aligned}
p(h)^{p'(k)} &= p(h)^{[k,1]} \\
&= 1p(h)^k 1 \\
&= p(h)^k \\
&= p(h^k)
\end{aligned}
$$

where the last equality follows from the fact that p is K-equivariant. $\square$

Definition 3.2.3. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of crossed modules. The **pushout** of H below f is the crossed module $[\mathsf{G}_2 \to \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2]$, denoted by $[\mathsf{E}^f : \partial^f]$.

Theorem 3.2.4. [24] Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of crossed modules. Then the natural morphism $\iota : [\mathsf{H} : \partial] \to [\mathsf{E}^f : \partial^f]$ induces a surjection $\pi_2(\mathsf{H}) \to \pi_2(\mathsf{E}^f)$ and an isomorphism $\pi_1(\mathsf{H}) \to \pi_1(\mathsf{E}^f)$. Moreover, there is a morphism $\rho : [\mathsf{E} : \partial^f] \to [\mathsf{G} : \delta]$ which completes the solid diagram

$$
\begin{tikzcd}
\mathsf{H}_2 \arrow[rr, "f_2"] \arrow[dr, "\iota_2"] \arrow[dd, "\partial"'] & & \mathsf{G}_2 \arrow[dl, equal] \arrow[dd, "\delta"] \\
& \mathsf{G}_2 \arrow[dd, "\partial^f", near start] & \\
\mathsf{H}_1 \arrow[rr, "f_1", crossing over] \arrow[dr, "\iota_1"'] & & \mathsf{G}_1 \\
& \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2 \arrow[ur, dashed, "\rho_1"'] &
\end{tikzcd}
$$

where $\rho_2 = 1_{\mathsf{G}_2}$.

3.3 Butterflies

We now apply the above results of pushouts to the factorization

$$\begin{tikzcd}
& \mathsf{Q} \arrow[d, "\nabla^f"] \arrow[ddl, "\simeq p"'] \arrow[ddr, "f"] & \\
& \mathsf{H} \times \mathsf{G} \arrow[dl, dashed, "\pi_1"] \arrow[dr, dashed, "\pi_2"'] & \\
\mathsf{H} \arrow[rr, dotted, "w"'] & & \mathsf{G}
\end{tikzcd}$$

Unfolding this pushout will give us a purely algebraic object called a *butterfly*; the name being owed to its shape as we will see shortly. Along with properly defined morphisms, butterflies form a groupoid which models the derived groupoid of crossed modules. In other words, studying morphisms of the homotopy category of 2-groups with respect to the Moerdijk-Svennson model structure and, equivalently, the morphisms of connected homotopy 2-types amounts to studying butterflies.

We give the following results of Behrang Noohi's work in [24] along with explicit proofs of any assumed details. These details will be useful to us in the generalization to n-groups.

Theorem 3.3.1. Let $p : [\mathsf{Q} : \xi] \to [\mathsf{H} : \partial]$ be an acyclic fibration and $f : [\mathsf{Q} : \xi] \to [\mathsf{G} : \delta]$ a morphism of crossed modules. Then the induced morphism $\iota : [\mathsf{Q} : \xi] \to [\mathsf{E}^{\nabla^f} : \xi^{\nabla^f}]$ is a weak equivalence.

Proof. By theorem 3.2.4, we only need to show that $\pi_2(\nabla)$ is injective. Since p is an acyclic fibration, $\mathsf{Q}_2 \cong \mathsf{Q}_1 \times_{\mathsf{H}_1} \mathsf{H}_2$ by Theorem 2.3.12. So we will view elements in Q_2 as elements in $\mathsf{Q}_1 \times_{\mathsf{H}_1} \mathsf{H}_2$. In this case, ξ is the projection on the first term and p_2 is the projection on the second term. Suppose $(x, y) \in \ker \xi$ and $\pi_2(\nabla)((x, y)) = (1, 1)$. Since $(x, y) \in \ker \xi$, $x = \xi((x, y)) = 1$. Since $\pi_2(\nabla) = \nabla_2$ on the kernel of ξ and $\pi_2(\nabla)((x, y)) = (1, 1)$, $p_2((x, y)) = 1$ and $f_2((x, y)) = 1$. Since $p_2((x, y)) = y$, $y = 1$. Thus, $(x, y) = (1, 1)$. Hence, $\pi_2(\nabla)$ is injective. $\square$

Using the 2-out-of-3 property of model categories and the fact that ι is clearly a fibration, we have the following fact.

Theorem 3.3.2. [24] Let $p : [\mathsf{Q} : \xi] \to [\mathsf{H} : \partial]$ be an acyclic fibration and $f : [\mathsf{Q} : \xi] \to [\mathsf{G} : \delta]$ a morphism of crossed modules. Then the induced map ρ in the factorization

$$\begin{tikzcd}
& \mathsf{Q} \arrow[d, "\nabla^f"] \arrow[ddl, "\simeq p"'] \arrow[ddr, "f"] & \\
& \mathsf{H} \times \mathsf{G} \arrow[dl, dashed, "\pi_1"] \arrow[dr, dashed, "\pi_2"'] & \\
\mathsf{H} \arrow[rr, dotted, "w"'] & & \mathsf{G}
\end{tikzcd}$$

is a trivial fibration.

Unfolding the Crossed module $\mathsf{E}^{\nabla f}$ along with the map ρ, we abstract the following definition.

Definition 3.3.3. Let $[\mathsf{H} : \partial]$ and $[\mathsf{G} : \delta]$ be crossed modules. A **butterfly** $B = (\mathsf{E}, p, f, \alpha, \beta)$ from H to G is a commutative diagram

$$\begin{tikzcd}
\mathsf{H}_2 \arrow[dd, "\partial"'] \arrow[dr, "\alpha"] & & \mathsf{G}_2 \arrow[dl, "\beta"'] \arrow[dd, "\delta"] \\
& \mathsf{E} \arrow[dl, "p"] \arrow[dr, "f"'] & \\
\mathsf{H}_1 & & \mathsf{G}_1
\end{tikzcd}$$

where both diagonal sequences are complexes and $\mathsf{G}_2 \to \mathsf{E} \to \mathsf{H}_1$ is short exact.

Definition 3.3.4. Let $[\mathsf{H} : \partial]$ and $[\mathsf{G} : \delta]$ be crossed modules. A morphism of butterflies

$$\Theta : (\mathsf{E}, p, f, \alpha, \beta) \longrightarrow (\mathsf{E}', p', f', \alpha', \beta')$$

is an group isomorphism $\Theta : \mathsf{E} \to \mathsf{E}'$ in which the diagram

$$\begin{tikzcd}
& & & \mathsf{G}_2 \arrow[dll] \arrow[dl] \arrow[dd, "\delta"] \\
\mathsf{H}_2 \arrow[r] \arrow[drr] \arrow[ddd, "\partial"'] & \mathsf{E} \arrow[dr, "\Theta"] \arrow[dddl, dotted] \arrow[ddrr, dotted] & & \\
& & \mathsf{E}' \arrow[dr] \arrow[ddll] & \\
& & & \mathsf{G}_1 \\
\mathsf{H}_1 & & &
\end{tikzcd} \tag{3.4}$$

commutes.

Since the morphism Θ is an isomorphism, Θ admits an inverse which satisfies the commutativity condition of the diagram 3.4. So the following result is clear.

Theorem 3.3.5. Butteflies from H to G as objects along with the morphisms defined above form a groupoid $\mathsf{B}(\mathsf{H}, \mathsf{G})$.

Now, we state the main theorem of the section.

Theorem 3.3.6. Let H and G be crossed modules. Then there is an equivalence of groupoids

$$\Omega : \underline{\mathbf{Rhom}}(\mathsf{H}, \mathsf{G}) \to \mathsf{B}(\mathsf{H}, \mathsf{G}).$$

Hence, we have found a purely algebraic model of the morphisms of the homotopy category of 2-groups with respect to the Moerdijk-Svensson model structure and, equivalently, the morphisms of homotopy 2-types.

CHAPTER 4

N-GROUPS AND N-TYPES

The relationship between connected homotopy types and ∞-groups is a much more recent result than for the 2-case. Furthermore, a complete algebraic model is still of current research interest. There is, however, a quasi-model given by a category with objects called crossed complexes along with a specified model structure. These objects and the model structure generalize the theory of crossed modules. We will give an overview of the relationship between connected homotopy types and ∞-groups, and show how crossed complexes fit into the picture. Before we formally introduce ∞-groups, we will define the more general notion of ∞-groupoids. Analogous to 2-groupoids, there are various degrees of strictness that we will also consider.

4.1 Infinity Groupoids

The idea of an *∞-groupoid* is more like a conceptual one; ideally, we should have objects, 1-morphisms, 2-morphisms, etc. Furthermore, these should satisfy associativity and unital conditions much like that of 1-groupoids and 2-groupoids. We will address two ideal notions of ∞-groupoids analogous to that of strict and weak 2-groupoids.

Using the formal definition of strict 2-groupoids and Remark 2.1.5, the definition of a *strict ∞-groupoid* (or commonly known as a *strict ω-groupoid*) below should be intuitive. We follow the approach of [2] in introducing the *globular category* and *globular sets* (or *∞-graphs*[1]) to help with the bookkeeping of k-morphisms. These are defined in a similar manner to simplicial sets (see [15]).

Definition 4.1.1. The **globular category O** is the category generated by the graph

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} 2 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} \cdots \underset{\tau_{k-1}}{\overset{\sigma_{k-1}}{\rightrightarrows}} k \underset{\tau_k}{\overset{\sigma_k}{\rightrightarrows}} k+1 \underset{\tau_{k+1}}{\overset{\sigma_{k+1}}{\rightrightarrows}} \cdots$$

satisfying the relations

$$\sigma_{k+1}\sigma_k = \tau_{k+1}\sigma_k$$

$$\sigma_{k+1}\tau_k = \tau_{k+1}\tau_k$$

[1]Both nomenclatures are well-devised. Globular sets are defined in a similar manner to simplicial sets with certain maps that define the n-hemispheres of n-globes. The alternative terminology, ∞-graphs, owes to the fact that an ∞-category can be generated by an ∞-graph much like a category can be generated by a graph.

for $k \geq 0$.

A **globular set** is a presheaf on $\mathbf{O}$.

Diagramatically, a globular set is a diagram

$$\cdots \underset{t_{k+1}}{\overset{s_{k+1}}{\rightrightarrows}} X_{k+1} \underset{t_k}{\overset{s_k}{\rightrightarrows}} X_k \underset{t_{k-1}}{\overset{s_{k-1}}{\rightrightarrows}} \cdots \underset{t_2}{\overset{s_2}{\rightrightarrows}} X_2 \underset{t_1}{\overset{s_1}{\rightrightarrows}} X_1 \underset{t_0}{\overset{s_0}{\rightrightarrows}} X_0$$

of sets and set maps satisfying the relations

$$s_k s_{k+1} = s_k t_{k+1}$$

$$t_k t_{k+1} = t_k s_{k+1}$$

for $k \geq 0$.

Considering s, t as source and target maps, we can view X_0 as the objects and X_k for $k \geq 1$ as the k-morphisms. Now, we just need to have a k-composition operation which satisfies associativity and unital laws, and is compatible with other dimensional compositions.

Let $s^i_j = s_j s_{j-1} \cdots s_{i-1}$ and $t^i_j = t_j t_{j-1} \cdots t_{i-1}$, and $X_i \times^{s,t}_{X_j} X_i$ be the fiber product of the maps s^i_j and t^i_j.

Definition 4.1.2. A **strict ω-category** $\mathcal{C}$ is a globular set with maps

$$\circ^i_j : X_i \times^{s,t}_{X_j} X_i \to X_i \qquad i > j \geq 0$$

$$1_k : X_{k-1} \to X_k \qquad k \geq 1$$

such that for $(a, b) \in X_i \times^{s,t}_{X_j} X_i$ and $x \in X_k$

$$s_{i-1}(a \circ^i_j b) = \begin{cases} s_{i-1}(b) & j = i-1 \\ s_{i-1}(a) \circ^{i-1}_j s_{i-1}(b) & j < i-1 \end{cases}$$

$$t_{i-1}(a \circ^i_j b) = \begin{cases} t_{i-1}(a) & j = i-1 \\ t_{i-1}(a) \circ^{i-1}_j t_{i-1}(b) & j < i-1 \end{cases}$$

and $s_k 1_{k+1}(x) = x = t_k 1_{k+1}(x)$ when $k \geq 0$, and the following conditions are satisfied.

Associativity: If $(a, b, c) \in X_i \times^{s,t}_{X_j} X_i \times^{s,t}_{X_j} X_i$ where $i > j \geq 0$, then

$$(a \circ^i_j b) \circ^i_j c = a \circ^i_j (b \circ^i_j c)$$

Interchange: If $(a, b, c, d) \in X_i \times^{s,t}_{X_j} X_i \times^{s,t}_{X_k} X_i \times^{s,t}_{X_j} X_i$ where $i > j > k \geq 0$, then

$$(a \circ^i_j b) \circ^i_k (c \circ^i_j d) = (a \circ^i_k c) \circ^i_j (b \circ^i_k d)$$

Units: If $x \in X_i$ with $i > j \geq 0$, then

$$x \circ_j^i 1_i^j \left(s_j^i(x)\right) = x = 1_i^j \left(t_j^i(x)\right) \circ_j^i x$$

where $1_i^j = 1_i 1_{i-1} \cdots 1_{j+1}$.

Functoriality of Units: If $(a, b) \in X_i \times_{X_j}^{s,t} X_i$ where $i > j \geq 0$, then

$$1_{a \circ_j^i b} = 1_a \circ_j^i 1_b$$

where $d1_i^j = 1_i 1_{i-1} \cdots 1_{j+1}$.

A **strict ω-functor** is a morphism of globular sets compatible with composition and units. Strict ω-categories along with strict ω-functors form a category denoted by ω**Cat**.

We will refer to $a \circ_j^i b$ as the (i, j)-composition of a with b and $1_k(x)$ as the k-identity map of x which is the k-identity map of x is a k-morphism. For a j-morphism x, we will write 1_x for the i-identity map $1_i^j(x) = 1_i 1_{i-1} \cdots 1_{j+1}(x)$ of x if clear from the context.

Definition 4.1.3. A **strict ω-groupoid** $\mathcal{G}$ is a strict ω-category in which every i-morphism a and $i > j \geq 0$ there is an i-morphism b such that $s_j(b) = t_j(a)$ and $t_j(b) = s_j(a)$, and

$$a \circ_j^i b = 1_{t_j^i(a)} \quad \text{and} \quad b \circ_j^i a = 1_{s_j^i(b)}$$

The i-morphism b is called a $\circ_j^i$**-inverse**. Strict ω-groupoids along with strict ω-functors form a category denoted by ω**Grpd**.

Of most importance will be the following objects.

Definition 4.1.4. A **strict ω-group** is a strict ω-groupoid with one object. A **strict n-group** is a strict ω-group in which X_k of the underlying globular set is empty for $k > n$. Strict ω-groups and strict n-groups along with strict ω-functors form a category denoted by ω**Grp** and n**Grp**, respectively.

We now give the weaker version of ∞-groupoids which, similarly to weak 2-groupoids, can be devised from topological spaces with the points of a space defining the objects, paths defining the 1-morphisms, homotopies defining the 2-morphisms, 2-homotopies defining the 3-morphisms, etc. In fact, Alexander Grothendieck postulated that the correct definition of ∞-groupoids should model topological spaces up to weak homotopy equivalence. Specifically, we have the following which should be viewed as a definition of the category of weak ∞-groupoids (or simply ∞-groupoids), denoted by ∞**Grpd**.

Homotopy Hypothesis 4.1.5. There is an equivalence of categories

$$\infty\mathbf{Grpd} \simeq Ho(\mathbf{Top}_Q).$$

Remark 4.1.6. There is a more general form of this theorem where the equivalence is of $(\infty, 1)$-categories, but we will refrain from explicitly stating that version since we are not formally introducing $(\infty, 1)$-categories in detail. In short, an $(\infty, 1)$ category is an ∞-category in which all k-morphisms are invertible for $k > 1$. For a more detailed excursion into $(\infty, 1)$-categories, one may look at [21].

Unfortunately, the associativity of homotopies only holds up to higher homotopies so strict ω-groupoids are to rigid. To parallel with the definition of homotopy 1-types and 2-types, we make the following definition.

Definition 4.1.7. The objects of $Ho(\mathbf{Top}_Q)$ are called **homotopy ∞-types** and the category $Ho(\mathbf{Top}_Q)$ is denoted by **HTyp**. The full subcategory of $Ho(\mathbf{Top}_Q)$ consisting of connected homotopy ∞-types is denoted by **cHTyp**.

The following theorem gives a more algebraic manifestation of ∞-groupoids. Explicitly, it gives an algebraic model of topological spaces up to weak homotopy equivalence i.e. homotopy ∞-types.

Theorem 4.1.8. [27] There is a Quillen equivalence

$$\mathbf{sSet}_Q \underset{sing(-)}{\overset{|-|}{\rightleftarrows}} \mathbf{Top}_Q$$

In particular, $Ho(\mathbf{sSet}_Q) \simeq Ho(\mathbf{Top}_Q)$.

Definition 4.1.9. A **weak ∞-groupoid** (or **∞-groupoid**[2]) is a fibrant-cofibrant object of $sSet_{Quillen}$.

Since every object in $\mathbf{sSet}_Q$ is cofibrant and the fibrant objects are precisely the Kan complexes, ∞-groupoids are the same as Kan complexes and the terminology is used interchangeably. In the next section, we will make absolutely clear the distinction between the strict and weak notions of ∞-groupoids (see Theorem 4.2.15).

[2] As the weaker definition of ∞-groupoid is preferred, this notion or anything equivalent to it is commonly referred to as ∞-groupoid.

4.2 Crossed Complexes

We now describe the algebraic structure of interest, crossed complexes. Traditionally, these were defined as a sequence of groupoids satisfying a list of properties. This definition can be found in numerous sources, including the book [6]. A much more functorial, but equivalent definition can be found in [2]. Our approach will be to use a more economical version which lies in between these two definitions.

Let $\mathbf{ChGrp}_+$ be the category of complexes with positive degrees beginning at 1 and the groups are not necessarily abelian.

Definition 4.2.1. A **crossed complex**[3] $[\mathsf{C}_1 : \mathsf{C} : \partial]$ is a groupoid C_1 fwith a set of objects C_0 and a functor $\mathsf{C} : \mathsf{C}_1 \to \mathbf{ChGrp}_+$ such that for all $x, y \in \mathsf{C}_0$ and $f \in \mathsf{C}_1(x, y)$:

1. $\partial(x)$ (or simply ∂) is the differential of $\mathsf{C}(x)$;
2. $\mathsf{C}(x)_1 = \mathsf{C}_1(x, x)$ and $\mathsf{C}(x)_n$ is abelian for $n \geq 3$;
3. $\mathsf{C}(f)_1 : \mathsf{C}(x)_1 \to \mathsf{C}(y)_1$ is conjugation by f;
4. if $a \in \mathsf{C}(x)_2$, then $\mathsf{C}(\partial_2(a))_k : \mathsf{C}(x)_k \to \mathsf{C}(x)_k$ is conjugation by a when $n = 2$ and is the identity map when $n \geq 3$.

A morphism of crossed complexes $[f_1 : f] : [\mathsf{C}_1 : \mathsf{C} : \partial] \to [\mathsf{D}_1 : \mathsf{D} : \delta]$ is a morphism of groupoids $f_1 : \mathsf{C}_1 \to \mathsf{D}_1$ and a natrual transformation $f : \mathsf{C} \to \mathsf{D}$. Crossed complexes along with these morphisms form a category denoted by $\mathbf{Xc}$.

We define further the maps ∂_0, ∂_1 to be the source and target maps of the groupoid C_1, respectively.

Remark 4.2.2. We prefer the above definition since it conceals much of the data of a crossed complex via the functor C. However, to see the resemblance between the definition we are using and the definition used by Ronald Brown and Marek Golasinski in [8], notice that given a crossed complex $[\mathsf{C} : \mathsf{C}_1 : \partial]$, the family of groups $\mathsf{C}_n := \{\mathsf{C}(x)_n\}_{x \in \mathsf{C}_0}$ forms a totally disconnected groupoid over C_0 for $n \geq 2$. Since they are totally disconnected over C_0, the family $\partial_n := \{\partial(x)_n\}_{x \in \mathsf{C}_0}$ of differentials define morphisms of groupoids for $n \geq 2$ giving the chain of groupoids:

$$\begin{tikzcd}
\ldots \arrow[r] & \mathsf{C}_5 \arrow[r, "\partial_5"] \arrow[d, shift left] \arrow[d, shift right] & \mathsf{C}_4 \arrow[r, "\partial_4"] \arrow[d, shift left] \arrow[d, shift right] & \mathsf{C}_3 \arrow[r, "\partial_4"] \arrow[d, shift left] \arrow[d, shift right] & \mathsf{C}_2 \arrow[r, "\partial_2"] \arrow[d, shift left] \arrow[d, shift right] & \mathsf{C}_1 \arrow[d, shift right, "\partial_0"'] \arrow[d, shift left, "\partial_1"] \\
\ldots \arrow[r, equal] & \mathsf{C}_0 \arrow[r, equal] & \mathsf{C}_0 \arrow[r, equal] & \mathsf{C}_0 \arrow[r, equal] & \mathsf{C}_0 \arrow[r, equal] & \mathsf{C}_0
\end{tikzcd}$$

[3]To be more explicit, these are commonly called *crossed complexes over a groupoid.*

The action of C_1 on C_k is induced by the functor C on morphisms of the groupoid C_1. The rest of the conditions are clear.

Definition 4.2.3. A crossed complex $[\mathsf{C}_1 : \mathsf{C} : \partial]$ in which C_1 is a groupoid with one object is a **reduced crossed complex**[4] and simply denoted by $[\mathsf{C} : \partial]$. A morphism of reduced crossed complexes $f : [\mathsf{C} : \partial] \to [\mathsf{D} : \delta]$ is a morphism of crossed complexes. Reduced crossed complexes along with morphisms form a full subcategory of **Xc** which is denoted by **xc**.

For reduced crossed complexes, we will usually ignore the object $*$ and visualize them diagrammatically as complexes

$$\ldots \longrightarrow \mathsf{C}_5 \xrightarrow{\partial_5} \mathsf{C}_4 \xrightarrow{\partial_4} \mathsf{C}_3 \xrightarrow{\partial_3} \mathsf{C}_2 \xrightarrow{\partial_2} \mathsf{C}_1 \tag{4.1}$$

Proposition 4.2.4. There are fully faithful functors

$$\mathbf{xm} \hookrightarrow \mathbf{xc} \hookrightarrow \mathbf{Xc}.$$

Proof. The second functor is clear. For the first, notice that for a reduced crossed complex $[\mathsf{C} : \partial]$ there is an action of C_1 on C_k induced by the functor C. Moreover, the conditions of crossed complexes imply that the map $\partial_2 : \mathsf{C}_2 \to \mathsf{C}_1$ is a crossed module. $\square$

In particular, the morphism $[\partial_2 : \mathsf{C}_2 \to \mathsf{C}_1]$ in Diagram 4.1 is a crossed module. Analogous to the relationship between crossed complexes over a group and over a groupoid, there are other crossed structures that generalize crossed modules called *crossed modules over a groupoid*. These are just crossed complexes where $\mathsf{C}(x)_k = 0$ for $k \geq 3$. We will not consider these in detail; however, for further reading one may consult [6].

We now begin describing some standard examples of crossed complexes.

Example 4.2.5. The initial object of **Xc** is the *empty crossed complex* $\emptyset$ where the set of objects is empty. The terminal object is the *unital crossed complex* 1 which is the reduced crossed complex with only trivial groups.

Remark 4.2.6. Notice that for the full subcategory of reduced crossed complexes **xc**, the empty crossed complex $\emptyset$ does not exist since it has no objects. In fact, the initial and final objects are both the unital crossed complex.

[4] Analogous to crossed complexes, these are commonly referred to as *crossed complexes over a group* which emphasizes the fact that a groupoid with one object may be viewed as a group.

The following crossed complex extends Example 2.2.4 of crossed modules. In fact, this example was the motivation for defining crossed complexes. First, we need some terminology.

Definition 4.2.7. A **filtered space** X_* is a space X and a sequence

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k \subset X_{k+1} \subset \cdots$$

of subspaces of X. A **filtered map** $f_* : X_* \to Y_*$ is a continuous map $f : X \to Y$ such that $f(X_k) \subset Y_k$ for $k \geq 0$. Filtered spaces along with filtered maps define a category **FTop**.

We call the filtered space X_* a *filtration* of X. For a simple example of a filtration, let X be a CW-complex. Then there is a natural filtration of X where X_n is defined to be the subspace of X composed of all k-cells for $k \leq n$.

Example 4.2.8. [6] Let X_* be a filtered space and $x \in X_0$. There is a complex

$$\cdots \longrightarrow \pi_{k+1}(X_{k+1}, X_k, x) \longrightarrow \pi_k(X_k, X_{k-1}, x) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_3(X_3, X_2, x) \longrightarrow \pi_2(X_2, X_1, x) \longrightarrow \pi_1(X_1, x)$$

where ∂_2 is the boundary map and for $k \geq 3$, ∂_k is the composition

$$\pi_k(X_k, X_{k-1}, x) \to \pi_{k-1}(X_{k-1}, x) \to \pi_{k-1}(X_{k-1}, X_{k-2}, x)$$

of the boundary map followed by the inclusion. Moreover, each map in the fundamental groupoid $\Pi_1 X_*$ induces a chain map. The details can then be checked that this defines a functor $\Pi X_* : \Pi_1 X_* \to \mathbf{ChGrp}_+$ and, further, that the triple $[\Pi_1 X_*, \Pi X_*, \partial]$ is a crossed complex. The resulting crossed complex is referred to as the *fundamental crossed complex* of the filtered space X_*. This construction defines a functor

$$[\Pi_1 : \Pi] : \mathbf{FTop} \to \mathbf{Xc}$$

which we will denote by Π for convenience.

We use the construction in the above example to define some simple, but important crossed complexes.

Example 4.2.9. Consider the n-sphere S^n for $n \geq 1$ with base-point denoted by 0. Viewing S^n as a CW-complex, the natural filtration S^n_* is defined by

$$S^n_k = \begin{cases} 0 & \text{for } \ k < n \\ S^n & \text{for } \ k \geq n \end{cases}$$

Then the fundamental crossed complex of S^n_* for $n \geq 1$ is the reduced crossed complex

$$\mathbb{S}^n : \ \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbb{Z} \longrightarrow \mathbf{1} \longrightarrow \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1}$$

where $\mathbb{Z}$ is in degree n and $\mathbf{1}$ is the trivial group.

Example 4.2.10. The 0-sphere S^0 viewed as a CW-complex is composed of two 0-cells which we will denote by 0 and 1. The natural filtration S^0_* is given by $S^0_k = 0 \cup 1$ for all k. The fundamental crossed complex of S^0_* can be visualized as the diagram

$$\mathbb{S}^0 : \begin{cases} \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} 0 \\ \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} 1 \end{cases}$$

Notice that $\mathbb{S}^0$ is not a reduced crossed complex.

Example 4.2.11. Consider the n-disk D^n for $n \geq 2$ with a base-point denoted by 0. Viewing D^n as a CW-complex, the natural filtration D^n_* is defined by

$$D^n_k = \begin{cases} 0 & \text{for} \quad k < n-1 \\ S^{n-1} & \text{for} \quad k = n-1 \\ D^n & \text{for} \quad k \geq n \end{cases}$$

The fundamental crossed complex of S^n_* for $n \geq 2$ is the reduced crossed complex

$$\mathbb{D}^n : \ \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbb{Z} \xrightarrow{1_{\mathbb{Z}}} \mathbb{Z} \longrightarrow \mathbf{1} \longrightarrow \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1}$$

where $\mathbb{Z}$ is in degree $n-1$ and n.

The next example will be important for defining a homotopy theory.

Example 4.2.12. The 1-disk D^1 (or alternatively, the unit interval I) viewed as a CW-complex is composed of two 0-cells which we will denote by 0 and 1, and a 1-cell connecting the two denoted by I. The 1-cell can be viewed as a path ι so it has a natural inverse ι^{-1}. Notice that the initial and final points of ι and ι^{-1} are different. The natural filtration D^1_* is given by

$$D^1_k = \begin{cases} 0 \cup 1 & \text{for} \quad k = 0 \\ I & \text{for} \quad k \geq 1 \end{cases}$$

The fundamental crossed complex of D^1_*, denoted by $\mathbb{D}^1$ (or alternatively, the *unit interval of crossed complexes* I), is similar to the diagram of $\mathbb{S}^0$; however, $\mathsf{I}(0,1)$ and $\mathsf{I}(1,0)$ are not the empty sets. Instead, $\mathsf{I}(0,1) = \{\iota\}$ and $\mathsf{I}(1,0) = \{\iota^{-1}\}$ with the relation that $\iota^{-1} \circ \iota$ and $\iota \circ \iota^{-1}$ are the identity maps of 0 and 1, respectively. The groupoid I_1 of the unit interval crossed complex defines the well-known *unit interval of groupoid* $\mathcal{I}$.

To begin our interlude into higher groups, we take a look at some examples of crossed complexes which derive from groupoids and 2-groupoids.

Example 4.2.13. Let $\mathcal{G}$ be a groupoid. We can easily define a crossed complex $[\mathcal{G} : \mathsf{G}]$ where the value of the functor $\mathsf{G} : \mathcal{G} \to \mathbf{ChGrp}_+$ at $x \in \mathcal{G}_0$ is the chain complex which is $\mathcal{G}(x, x)$ in the first degree and the trivial group in all other degrees with ∂ being the trivial map. More explicitly, we can view this crossed complex as

$$\ldots \xrightarrow{\partial} \{\mathbf{1}\}_{\mathcal{G}_0} \xrightarrow{\partial} \{\mathbf{1}\}_{\mathcal{G}_0} \xrightarrow{\partial} \{\mathbf{1}\}_{\mathcal{G}_0} \xrightarrow{\partial} \{\mathbf{1}\}_{\mathcal{G}_0} \xrightarrow{\partial} \mathcal{G} \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \mathcal{G}_0$$

where $\{\mathbf{1}\}_{\mathcal{G}_0}$ is the family of trivial groups over $\mathcal{G}_0$. Of course, the groupoid can be recovered from the crossed complex $[\mathcal{G} : \mathsf{G}]$ since it is $\mathcal{G}$.

Example 4.2.14. Given a 2-groupoid $\mathcal{G}$, we can define a crossed complex $[\mathcal{G}_1 : \mathsf{G}]$ where $\mathcal{G}_1$ is the underlying groupoid of $\mathcal{G}$ composed of objects and 1-morphisms. The functor $\mathsf{G} : \mathcal{G}_1 \to \mathbf{ChGrp}_+$ is defined by sending each object $x \in \mathcal{G}_0$ to the chain complex which is $\mathcal{G}_1(x, x)$ in the first degree, the union of sets of 2-morphisms $\bigcup_\lambda \mathcal{G}(1_x, \lambda)$ where $\lambda \in \mathcal{G}_1(x, x)$ in the second degree with the group structure defined by horizontal composition, and the trivial group in all other degrees. Define the differential by $\partial_2(\alpha) = t(\alpha)$ and all others being the trivial map. More explicitly, we can view this crossed complex as the following chain of families of groups over $\mathcal{G}_0$:

$$\ldots \xrightarrow{\partial_5} \{\mathbf{1}\}_{\mathcal{G}_0} \xrightarrow{\partial_4} \{\mathbf{1}\}_{\mathcal{G}_0} \xrightarrow{\partial_3} \{\textstyle\bigcup_\lambda \mathcal{G}(1_x, \lambda)\}_{\mathcal{G}_0} \xrightarrow{\partial_2} \mathcal{G}_1 \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \mathcal{G}_0$$

Theorem 4.2.15. [9] The category of crossed complexes is equivalent to the category of strict ω-groupoids. Moreover, the category of reduced crossed complexes is equivalent to the category of strict ∞-groups.

Due to the above theorem, we will follow the notation for crossed modules and use the letters G, H of group theory to denote reduced crossed complexes.

Dold-Kan Correspondence 4.2.16. [12] There is a functor called the *Moore chain complex functor* N (or the *normalized chain complex functor*) which induces the equivalence

$$\mathbf{sAb} \underset{\simeq}{\xrightarrow{N}} \mathbf{Ch}_{\geq 0}(R)$$

where $\mathbf{sAb}$ is the category of simplicial abelian groups. Moreover, under this correspondence simplicial homotopies correspond to chain homotopies and simplicial homotopy groups correspond to homology groups.

Remark 4.2.17. The Dold-Kan Correspondence actually applies to simplicial objects and bounded chain complexes in any abelian category; however, for ours interests we stated the specific result.

The following theorem which was proved in the thesis [3] of Nick Ashley is the main result of the section.

Dold-Kan Extension 4.2.18. There is a fully faithful functor

$$\mathbf{Xc} \hookrightarrow \mathbf{sGrpd}\ .$$

which restricts to a fully faithful functor

$$\mathbf{xc} \hookrightarrow \mathbf{sGrp}\ .$$

The categorical image of the functors above are the full subcategories of **sGrpd** and **sGrp** consisting of groupoid T-complexes and group T-complexes, respectively. Both of these are discussed further in [3]. Our main interest will be in the reduced crossed complex case which can be clarified by the following results.

Theorem 4.2.19. [15] There is a Quillen equivalence

$$\mathbf{sGrp}_Q \underset{}{\overset{\overline{W}}{\rightleftarrows}} \mathbf{sSet}_Q^{red}$$

where $\mathbf{sSet}^{red}$ is the category of simplicial sets with only one 0-cell.

Corollary 4.2.20. There is an equivalence of categories

$$Ho(\mathbf{sGrp}_Q) \simeq Ho(\mathbf{sSet}_Q^{red}).$$

This corollary along with the fact below explain why we would like to study reduced crossed complexes.

Theorem 4.2.21. [15] There is a fully faithful functor

$$Ho(\mathbf{sSet}_Q^{red}) \hookrightarrow Ho(\mathbf{sSet}_Q)$$

which restricts to the equivalence

$$Ho(\mathbf{sSet}_Q^{red}) \simeq \mathbf{cHTyp}.$$

As *weak* ∞-groupoids can be viewed as categorical models of topological spaces, the objects being the points, 1-morphisms the paths, 2-morphisms the homotopies between paths, etc., the above equivalences show that crossed complexes are algebraic models of a particular class of spaces. Thus, we now focus our attention completely on crossed complexes, beginning with a short exposition on algebraic and topological constructions.

4.3 Monoidal Structure on Crossed Complexes

We now describe the closed monoidal structure on **Xc** introduced in [10]. As described by Brown and Higgins, the tensor product of crossed complexes could have been purely defined by passing the tensor of strict ω-groupoids through the equivalence from Theorem 4.2.15. Fortunately, the focus of this paper was to give an explicit description completely formulated in terms of crossed complexes which we now describe.

Definition 4.3.1. The **tensor product** of the crossed complexes C and D is the crossed complex $\mathsf{C} \otimes \mathsf{D}$ where the groupoid $(\mathsf{C} \otimes \mathsf{D})_k$ over $\mathsf{C}_0 \times \mathsf{D}_0$ is generated by the elements $c_i \otimes d_j$ where $i + j = k$, $c_i \in \mathsf{C}_i$ and $d_j \in \mathsf{D}_j$, and $\mathsf{C} \otimes \mathsf{D}$ satisfies the following laws.

Law of Base Points:

$$s(c_0 \otimes d_1) = c_0 \otimes sd_1$$
$$s(c_1 \otimes d_0) = sc_1 \otimes d_0$$

$$\begin{aligned} t(c_0 \otimes d_j) &= c_0 \otimes td_j & j \geq 1 \\ t(c_i \otimes d_0) &= sc_i \otimes d_0 & i \geq 1 \\ t(c_i \otimes d_j) &= tc_i \otimes td_j & i, j \geq 1 \end{aligned}$$

Law of Differentials:

$$\begin{aligned} \delta(c_1 \otimes d_1) &= (tc_1 \otimes d_1)^{-1} (c_1 \otimes sd_1)^{-1} (sc_1 \otimes d_1) (c_1 \otimes td_1) \\ \delta(c_0 \otimes d_j) &= c_0 \otimes \delta d_j & \text{if} j \geq 2 \\ \delta(c_1 \otimes d_j) &= (c_1 \otimes \delta d_j)^{-1} (tc_1 \otimes d_j)^{-1} (sc_1 \otimes d_j)^{c_1 \otimes td_j} & \text{if} j \geq 2 \\ \delta(c_i \otimes d_0) &= \delta c_i \otimes d_0 & \text{if } i \geq 2 \\ \delta(c_i \otimes d_1) &= (c_i \otimes td_1)^{(-1)^{i+1}} (c_i \otimes sd_1)^{(-1)^i (tc_i \otimes d_1)} (\delta c_i \otimes d_1) & \text{if } i \geq 2 \\ \delta(c_i \otimes d_j) &= (\delta c_i \otimes d_j)(c_i \otimes \delta d_j)^{(-1)^i} & \text{if } i \geq 2, j \geq 2 \end{aligned}$$

Distributive Laws:

$$(c_i \cdot c_i') \otimes d_j = \begin{cases} c_i \otimes b + c_i' \otimes b & \text{if } i \geq 1, j = 0 \text{ or if } i \geq 2, j \geq 1 \\ c_i' \otimes b + (c_i \otimes b)^{c_i' \otimes tb} & \text{if } i = 1, j \geq 1 \end{cases} \tag{4.2}$$

$$c_i \otimes (d_j \cdot d_j') = \begin{cases} c_i \otimes d_j + c_i \otimes d_j' & \text{if } i = 0, j \geq 1 \text{ or if } i \geq 1, j \geq 2 \\ (c_i \otimes d_j)^{tc_i \otimes d_j'} + c_i \otimes d_j' & \text{if } i \geq 1, j = 1 \end{cases} \tag{4.3}$$

Law of Actions

$$
\begin{aligned}
c_i^{c_1} \otimes d_j &= (c_i \otimes d_j)^{c_1 \otimes t d_j} && \text{if } i \geq 2,\ j \geq 0 \\
c_i \otimes d_j^{d_1} &= (c_i \otimes d_j)^{t c_i \otimes d_1} && \text{if } i \geq 0,\ j \geq 2
\end{aligned}
$$

Remark 4.3.2. The tensor was equivalently defined by Tonk in [28] as a map into *double complexes* composed with a *total functor*, analogous to chain complexes.

Two crossed complexes constructed using the tensor product which will be useful later are the cylinder and cone of a crossed complex. These are given in [11] along with an explicit definition of the lower dimensions in terms of generators.

Example 4.3.3. The *cylinder* of a crossed complex C, denoted by $cyl(\mathsf{C})$, is the crossed complex $\mathsf{C} \otimes \mathsf{I}$ where I is as defined in example 4.2.12. Also, we have the maps

$$i_0 : \mathsf{C} \to \mathsf{C} \otimes \mathsf{I}$$

$$i_1 : \mathsf{C} \to \mathsf{C} \otimes \mathsf{I}$$

defined on generators by $i_0(c_n) = c \otimes 0$ and $i_1(c_n) = c_n \otimes 1$, respectively.

Again, recall that the unital crossed complex is the reduced crossed complex with only trivial groups. This crossed complex acts as unity with respect to $\otimes$.

Example 4.3.4. The *cone* of a crossed complex C, $cone(\mathsf{C})$, is the crossed complex defined by the pushout

$$
\begin{tikzcd}
1 \otimes \mathsf{C} \arrow[r] \arrow[d] & 1 \arrow[d] \\
\mathsf{I} \otimes \mathsf{C} \arrow[r] & cone(\mathsf{C})
\end{tikzcd}
$$

where the map $1 \to cone(\mathsf{C})$ gives the vertex of the cone.

Continuing with the monoidal structure on $\mathbf{Xc}$, we have the next two properties of the tensor product given in [10].

Theorem 4.3.5. For any $\mathsf{A}, \mathsf{B}, \mathsf{C} \in \mathbf{Xc}$, there are isomorphisms of crossed complexes

1. $\mathsf{A} \otimes (\mathsf{B} \otimes \mathsf{C}) \cong (\mathsf{A} \otimes \mathsf{B}) \otimes \mathsf{C}$
2. $\mathsf{A} \otimes \mathsf{B} \cong \mathsf{B} \otimes \mathsf{A}$.

Moreover, $\otimes$ and 1 satisfy associativity and unital identities of monoidal categories (see [5]) so we have the following.

Theorem 4.3.6. [10] The category of crossed complexes $(\mathbf{Xc}, \otimes, 1)$ is a symmetric monoidal category.

Now we give a description of the internal hom of two crossed complexes.

Definition 4.3.7. Let $\mathsf{C}, \mathsf{D} \in \mathbf{Xc}$. A **$m$-fold left homotopy** from C to D is an ordered pair (g, ϕ^m) where $g \in \mathbf{Xc}(\mathsf{C}, \mathsf{D})$ and ϕ^m is a family of degree m morphisms $\phi^m_n : \mathsf{C_n} \to \mathsf{D}_{m+n}$ satisfying:

Base Law:

$$\begin{aligned} t\phi^m_0(c_0) &= g_0(c_0) \\ t\phi^m_n(c_n) &= tg_n(c_n) && \text{if } n \geq 1 \end{aligned}$$

Preservation Law:

$$\begin{aligned} \phi^m_1(c_1 \cdot c'_1) &= \phi^m_1(c_1)^{g_1(c'_1)} \cdot \phi^m_1(c'_1) \\ \phi^m_n(c_n \cdot c'_n) &= \phi^m_n(c_n) \cdot \phi^m_n(c'_n) && \text{if } n \geq 2 \end{aligned}$$

Equivariance Law:

$$\phi^m_n(c_n^{c_1}) = \phi^m_n(c_n)^{g_1(c_1)} \qquad \text{if } n \geq 2$$

If $\mathsf{C}, \mathsf{D} \in \mathbf{xc}$, then a **pointed m-fold left homotopy** is a m-fold left homotopy in which $\phi^m_0(*) = 1$ where $*$ is the only object.

The morphism g in the above definition is referred to as the *base morphism of the homotopy.*

Remark 4.3.8. There is a similar definition for m-fold right homotopies which changes the preservation law for case $n = 1$ to $h(c') \cdot h(c)^{f(c')}$. Notice that this difference is only a matter of order which becomes immaterial in degrees $n \geq 2$ since all groups are abelian. Thus, the notion of m-fold left and right homotopies concur for $n \geq 2$.

Definition 4.3.9. Let $\mathsf{C}, \mathsf{D} \in \mathbf{Xc}$. The **internal hom** from C to D is the crossed complex $\mathbf{XC}(\mathsf{C}, \mathsf{D})$ where $\mathbf{XC}(\mathsf{C}, \mathsf{D})_0 = \mathbf{Xc}(\mathsf{C}, \mathsf{D})$ and $\mathbf{XC}(\mathsf{C}, \mathsf{D})_k$ is the set of k-fold left homotopies from C to D.

The next theorem relating the tensor with the internal hom, finalizes the fact that $(\mathbf{Xc}, \otimes, 1)$ is a *closed symmetric monoidal category* (see [5]).

Theorem 4.3.10. [10] For any $\mathsf{B} \in \mathbf{Xc}$, there is an adjunction

$$- \otimes \mathsf{B} : \mathbf{Xc} \rightleftarrows \mathbf{Xc} : \mathbf{Xc}(\mathsf{B}, -)$$

Moreover, there is an isomorphism of crossed complexes

$$\mathbf{XC}(\mathsf{A} \otimes \mathsf{B}, \mathsf{C}) \cong \mathbf{XC}\left(\mathsf{A}, \mathbf{Xc}(\mathsf{B}, \mathsf{C})\right)$$

Remark 4.3.11. The first result of the theorem can actually be seen as a corollary of the second as the zero elements of the internal hom are the morphisms.

4.4 Homotopy Theory for Crossed Complexes

Using the tensor product and the unit interval, we can now define a intuitive notion of homotopy between two morphisms of crossed complexes.

Definition 4.4.1. Let $f, g : \mathsf{C} \to \mathsf{D}$ be two morphisms in $\mathbf{Xc}$. A **homotopy** from f to g, denoted by $h : f \simeq g$, is a morphism of crossed complexes $h : \mathsf{C} \otimes \mathsf{I} \to \mathsf{D}$ which makes the diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[d, "i_0"'] \arrow[dr, "f"] & \\
\mathsf{C} \otimes \mathsf{I} \arrow[r, "h"] & \mathsf{D} \\
\mathsf{C} \arrow[u, "i_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commute.

The following fact shows that homotopies as defined above are in fact related to 1-fold left homotopies which was proven by Andrew Tonk in [28].

Theorem 4.4.2. Let $f, g : [\mathsf{C}_1 : \mathsf{C} : \partial] \to [\mathsf{D}_1 : \mathsf{D} : \delta]$ be two morphisms in $\mathbf{Xc}$. A homotopy $h : f \simeq g$ is equivalent to defining a left homotopy $(g, \phi_n : \mathsf{C}_n \to \mathsf{D}_{n+1})$. Moreover, f is completely determined by the algebraic equations

$$f_0(c_0) = s(\phi_0(c_0)) \tag{4.4}$$

$$f_1(c_1) = \phi_0(s(c_1))g_1(c_1)\delta_2(\phi_1(c_1))\phi_0(t(c_1))^{-1} \tag{4.5}$$

$$f_n(c_n)^{\phi_0(t(c_n))} = g_n(c_n)\delta_{n+1}(\phi_n(c_n))\phi_{n-1}(\partial_n(c_n)) \qquad \text{for } n \geq 2 \tag{4.6}$$

A strict ω-group viewed as a group object in the category of strict ∞-groupoids can be delooped to give a pointed connected strict ω-groupoid without losing any information; however, the pointed detail must not be forgotten. So to properly study the homotopy theory of an ∞-group through reduced crossed complexes, we need a notion of pointed homotopy.

Definition 4.4.3. Let $f, g : [\mathsf{C} : \partial] \to [\mathsf{D} : \delta]$ be morphisms in $\mathbf{xc}$. A **pointed homotopy** $h : f \simeq g$ is a homotopy which can be described by a pointed 1-fold left homotopy.

Explicitly, a homotopy $h : f \simeq g$ is pointed if and only if there exits a pointed left-homotopy (g, ϕ) in which f is determined by

$$f_1(c_1) = g_1(c_1)\delta_2(\phi_1(c_1)) \qquad \text{for } n \geq 1$$
$$f_k(c_k) = g_k(c_k)\delta_{k+1}(\phi_k(c_k))\phi_{k-1}(\partial_k(c_k)) \qquad \text{for } k > 1$$

Remark 4.4.4. We have chosen to completely ignore the single objects of both C and D as it is clear how the homotopy acts on them. It should also be noted that if we consider reduced crossed complexes in which the degree one and two groups are abelian, this notion of homotopy aligns with the usual chain complex homotopy.

In relation to spaces, crossed complexes naturally carry a model structure which was derived from the model structure on strict ω-groupoids and the equivalence above by Brown and Golasinski in [8]. As one might suspect, this model structure is defined in terms of an algebraic analog of homotopy groups much like chain complexes.

Definition 4.4.5. Let $[\mathsf{C}_1 : \mathsf{C} : \partial]$ be a crossed complex. The **connected components** is the set $\pi_0(\mathsf{C}_1)$ of connected components of the groupoid C_1. For each $x \in \mathsf{C}_0$, the **fundamental homotopy group** of C at x is the group $\pi_1(\mathsf{C}, x) = coker[\delta_2 : \mathsf{C}(x)_2 \to \mathsf{C}(x)_1]$. The n**-homotopy group** of C at x is the group $\pi_n(\mathsf{C}, x) = H_n(\mathsf{C}(x))$.

Theorem 4.4.6. [8] There exists a model structure on $\mathbf{Xc}$ where a morphism $f : \mathsf{C} \to \mathsf{D}$ in $\mathbf{Xc}$ is a *weak equivalence* if the induced set map of connected components is a bijection and the induced maps of the fundamental homotopy group and n-homotopy groups are isomorphisms; a *fibration* if $f_1 : \mathsf{C}_1 \to \mathsf{D}_1$ is a fibration of groupoids[5] and $f_n : \mathsf{C}(x)_n \to \mathsf{D}(f_1(x))_n$ is surjective for $n \geq 2$ and all $x \in \mathsf{C}_0$.

Cofibrations for the model structure above are precisely the morphisms which have the left lifting property with respect to trivial fibrations (see Theorem A.1.13).

Since the unique map $\mathsf{C} \to 1$ is clearly a fibration of crossed complexes for any $\mathsf{C} \in \mathbf{Xc}$, every crossed complex is fibrant; however, not every crossed complex is cofibrant.

To help better understand cofibrant objects and cofibrations in general, the following theorem gives a nice characterization of trivial fibrations which was described by Ronald Brown and Marek Golansinski in [8].

[5] There is a detailed exposition on fibrations of groupoids in [7].

Theorem 4.4.7. A morphism $f : [\mathsf{C}_1 : \mathsf{C} : \partial] \to [\mathsf{D}_1 : \mathsf{D} : \delta]$ of crossed complexes is a trivial fibration if and only if

1. the set function $f_0 : \mathsf{C}_0 \to \mathsf{D}_0$ is surjective;
2. the functor $f_1 : \mathsf{C}_1 \to \mathsf{D}_1$ is full;
3. for every $x \in \mathsf{C}_0$, the induced maps

   $$\mathsf{C}(x)_n \to \ker \partial_{n-1}(x) \times_{\ker \delta_{n-1}(x)} \mathsf{D}(x)_n$$

   are surjective for $n \geq 2$.

Remark 4.4.8. For $n = 2$ in the above theorem, $\ker \partial_{n-1}(x)$ and $\ker \delta_{n-1}(x)$ are the equalizers of the diagrams

$$\mathsf{C}(x)_1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathsf{C}(x)_0 \qquad\qquad \mathsf{D}(x)_1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathsf{D}(x)_0$$

respectively. These are precisely $\mathsf{C}(x)_1$ and $\mathsf{D}(x)_1$, respectively.

The Whitehead Theorem for Crossed Complexes 4.4.9. [8] Let

$$f : [\mathsf{C}_1 : \mathsf{C} : \partial] \to [\mathsf{D}_1 : \mathsf{D} : \delta]$$

be a morphism of cofibrant objects in **Xc**. If f is a weak equivalence, then it is a homotopy equivalence.

Remark 4.4.10. The theorem above is an important result since the homotopy equivalence in the statement is with respect to homotopy of crossed complexes as compared to homotopy of model categories. Since every crossed complex is fibrant, it is a standard result of model categories that, for morphisms between cofibrant objects in **Xc**, weak equivalence is equivalent to homotopy equivalence of model categories (see the Whitehead Theorem for Model Categories A.2.26).

Corollary 4.4.11. Let

$$f : [\mathsf{C}_1 : \mathsf{C} : \partial] \to [\mathsf{D}_1 : \mathsf{D} : \delta]$$

be a morphism of cofibrant objects in **Xc**. The map f is a weak equivalence if and only if it is a homotopy equivalence.

Proof. Since homotopies of crossed complexes can be replaced by 1-fold left homotopies, homotopy equivalence of crossed complexes can be expressed using 1-fold left homotopies. Using the similarity between 1-fold left homotopies and chain maps of chain complexes, homotopy equivalence induces isomorphisms on homotopy groups of crossed complexes. □

4.5 n-Crossed Complexes

We would like to adapt the work of Ettore Aldrovandi and Behrang Noohi on butterflies from crossed modules to crossed complexes. There are a couple difficulties in the adaptation. In particular, crossed complexes are over a groupoid while crossed modules are over a group. Further, even if we restrict to reduced crossed complexes, we are now considering infinite complexes of groups instead of a group homomorphism; not to mention, these chains are part non-abelian. So to ease into the problem, we will be only considering reduced crossed complexes that are trivial beyond a certain degree. However, we will use this section to introduce these crossed complexes without stipulating that they are reduced.

Let $n\mathbf{ChGrp}_+$ be the category of complexes of length n with positive degrees beginning at 1 and the groups are not necessarily abelian.

Definition 4.5.1. Let $n \geq 1$. A n**-crossed complex** is a crossed complex $[\mathsf{C}_1 : \mathsf{C} : \partial]$ in which $\mathsf{C} : \mathsf{C}_1 \to n\mathbf{ChGrp}_+$.

Considering n-crossed complexes along with morphisms of crossed complexes form a category denoted by $n\mathbf{Xc}$. The following constructions and more details on n-crossed complexes can be found in [6].

Theorem 4.5.2. Let $[\mathsf{C}_1 : \mathsf{C} : \partial] \in \mathbf{Xc}$. The following are functors.

The *n-truncation tr_n* $: \mathbf{Xc} \to n\mathbf{Xc}$ assigns to $[\mathsf{C}_1 : \mathsf{C} : \partial]$ the n-crossed complex $[\mathsf{C}_1 : tr_n(\mathsf{C}) : tr_n(\partial)]$ where the complex over each object $x \in \mathsf{C}_0$ is of the form:

$$\mathsf{C}_n \xrightarrow{\partial_n} \mathsf{C}_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_3} \mathsf{C}_2 \xrightarrow{p_2} \mathsf{C}_1$$

The *n-cotruncation $cotr_n$* $: \mathbf{Xc} \to n\mathbf{Xc}$ assigns to $[\mathsf{C} : \mathsf{C}_1 : \partial]$ the n-crossed complex $[\mathsf{C}_1 : cotr_n(\mathsf{C}) : cotr_n(\partial)]$ where the complex over each object $x \in \mathsf{C}_0$ is of the form:

$$\mathsf{C}_n/\partial_{n+1} \xrightarrow{\overline{\partial}_n} \mathsf{C}_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_3} \mathsf{C}_2 \xrightarrow{\partial_2} \mathsf{C}_1$$

where we define $\mathsf{C}_n/\partial_{n+1} = \mathsf{C}_n/\mathrm{Im}(\partial_{n+1})$.

Theorem 4.5.3. Let $[\mathsf{C}_1 : \mathsf{C} : \partial] \in n\mathbf{Xc}$. The following are functors.

The *n-skeleton sk^n* $: n\mathbf{Xc} \to \mathbf{Xc}$ assigns to $[\mathsf{C}_1 : \mathsf{C} : \partial]$ the crossed complex $[\mathsf{C}_1 : sk^n(\mathsf{C}) : sk^n(\partial)]$ where the complex over each object $x \in \mathsf{C}_0$ is of the form:

$$\dots \longrightarrow 0 \longrightarrow \mathsf{C}_n \xrightarrow{\partial_n} \mathsf{C}_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_3} \mathsf{C}_2 \xrightarrow{\partial_2} \mathsf{C}_1$$

The *n-coskeleton* $cosk^n : n\mathbf{Xc} \to \mathbf{Xc}$ assigns to $[\mathsf{C}_1 : \mathsf{C} : \partial]$ the crossed complex $[\mathsf{C}_1 : cosk^n(\mathsf{C}) : cosk^n(\partial)]$ where the complex over each object $x \in \mathsf{C}_0$ is of the form:

$$\ldots \longrightarrow 0 \longrightarrow \ker \delta_n \xrightarrow{i} \mathsf{C}_n \xrightarrow{\partial_n} \mathsf{C}_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_3} \mathsf{C}_2 \xrightarrow{\partial_2} \mathsf{C}_1$$

Remark 4.5.4. The n-skeleton functor allows us to consider $n\mathbf{Xc}$ as a full subcategory of $\mathbf{Xc}$. So we will view an object of $n\mathbf{Xc}$ as a crossed complex which is trivial in degrees greater than n.

Theorem 4.5.5. [6] The functors $tr_n, cotr_n, sk^n, cosk^n$ satisfy the adjoint[6] diagrams

$$tr_n : \mathbf{Xc} \rightleftarrows n\mathbf{Xc} : cosk^n$$

$$cotr_n : \mathbf{Xc} \rightleftarrows n\mathbf{Xc} : sk^n$$

$$sk^n : n\mathbf{Xc} \rightleftarrows \mathbf{Xc} : tr_n$$

Two important functors are the compositions $Cosk^n = cosk^n \circ tr_n : \mathbf{Xc} \to \mathbf{Xc}$ and $Sk^n = sk^n \circ tr_n : \mathbf{Xc} \to \mathbf{Xc}$. Now, we can give an analog of the theorem 4.4.7 for $n\mathbf{Xc}$.

Proposition 4.5.6. A morphism $f : \mathsf{C} \to \mathsf{D}$ of n-crossed complexes is a trivial fibration if and only if

1. the set function $f_0 : \mathsf{C}_0 \to \mathsf{D}_0$ is surjective;
2. the functor $f_1 : \mathsf{C}_1 \to \mathsf{D}_1$ is full;
3. for every $x \in \mathsf{C}_0$, the induced maps

$$\mathsf{C}(x)_k \to Cosk^{k-1}(\mathsf{C}(x))_k \times_{Cosk^{k-1}(\mathsf{D}(x))_k} \mathsf{D}(x)_k \tag{4.7}$$

are surjective for $2 \leq k < n$ and an isomorphism for $k = n$.

Proof. For a trivial fibration of n-crossed complexes, everything is clear from theorem 4.4.7 except for the injection in degree n. Explicitly, we want to show that for each $x \in \mathsf{C}_0$ the universal group homomorphism

$$u_n : \mathsf{C}(x)_n \to Cosk^{n-1}(\mathsf{C}(x))_n \times_{Cosk^{n-1}(\mathsf{D}(x))_n} \mathsf{D}(x)_n \tag{4.8}$$

is injective and thus an isomorphism for each $x \in \mathsf{C}_0$. Fixing an arbitrary $x \in \mathsf{C}_0$, all groups here forth will be assumed to be over x.

[6]For the adjoint diagrams, left adjoints are written on the left and right adjoints are written on the right.

Recall that for $n \geq 2$ the morphism 4.8 is guaranteed by the diagram

$$\begin{tikzcd}
 & & \mathsf{D}_n \arrow[dr, "\delta_n"] & \\
\mathsf{C}_n \arrow[r, dashed, "u_n"] \arrow[urr, bend left, "f_n"] \arrow[drr, bend right, "\partial_n"'] & \ker \partial_{n-1} \times_{\ker \delta_{n-1}} \mathsf{D}_n \arrow[ur, "\pi_2"] \arrow[dr, "\pi_1"'] & & \ker \delta_{n-1} \\
 & & \ker \partial_{n-1} \arrow[ur, "\overline{f}_{n-1}"'] &
\end{tikzcd}$$

where $\overline{f}_{n-1}$ is the restriction to kernels and ∂_n, δ_n are understood to be the usual maps with restricted codomains. The latter morphisms are allowed since ∂, δ are chain maps. Note that from remark 4.4.8, for the case when $n = 2$ we have that

$$\ker \partial_{n-1} = Cosk^{n-1}(\mathsf{C})_n = \mathsf{C}_1$$

and

$$\ker \delta_{n-1} = Cosk^{n-1}(\mathsf{D})_n = \mathsf{D}_1$$

Since f is a weak equivalence, the induced morphisms of homotopy groups are all isomorphisms. Since and $\mathsf{C}_k = \mathsf{D}_k = 0$ for $k > n$, the nth morphism of homotopy groups is in fact $\overline{f}_n$. In general, we have the diagram

$$\begin{tikzcd}
\ker \partial_n \arrow[r] \arrow[d] \arrow[rr, bend left, "\overline{f}_n"] & \ker \pi_1 \arrow[r] \arrow[d] & \ker \delta_n \arrow[d] \\
\mathsf{C}_n \arrow[r, "u_n"] \arrow[d, "\partial_n"'] & \ker \partial_{n-1} \times_{\ker \delta_{n-1}} \mathsf{D}_n \arrow[r] \arrow[d, "i \circ \pi_1"] & \mathsf{D}_n \arrow[d, "\delta_n"] \\
\mathsf{C}_{n-1} \arrow[r, equal] \arrow[d] & \mathsf{C}_{n-1} \arrow[r] \arrow[d] & \mathsf{D}_{n-1} \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

where $i \circ \pi_1$ is the projection on the first term followed by the inclusion into C_{n-1}. Suppose $a \in \mathsf{C}_n$ such that $a \in \ker u_n$. Then

$$(1, 1) = u_n(a) = (\partial_n(a), f_n(a))$$

So $a \in \ker \partial_n$. Thus, $\overline{f}_n(a)$ is defined and in fact $\overline{f}_n(a) = f_n(a) = 1$. Since $\overline{f}_n$ is injective, $a = 1$. Thus, u_n is injective. The converse is immediate from theorem 4.4.7. $\square$

Definition 4.5.7. The **weak n-replacement** of a crossed complex $[\mathsf{C}_1 : \mathsf{C} : \partial]$ is the crossed complex $[\mathsf{C}_1 : r^n(\mathsf{C}) : r^n(\partial)]$ defined by $r^n(\mathsf{C}) = sk^n(cotr_n(\mathsf{C}))$ and $r^n(\partial) = sk^n(cotr_n(\partial))$.

Remark 4.5.8. For a n-crossed complex $[\mathsf{C}_1 : \mathsf{C} : \partial]$ viewed as a crossed complex, the weak n-replacement of C is simply C i.e. $r^n(\mathsf{C}) = \mathsf{C}$.

Proposition 4.5.9. Let $[\mathsf{C}_1 : \mathsf{C} : \partial]$ be a crossed complex. Then there is a canonical morphism $\mathsf{C} \to r^n(\mathsf{C})$ of crossed complexes.

Proof. The morphism $\mathsf{C} \to r^n(\mathsf{C})$ is induced by the quotient map $\mathsf{C}_n \to \mathsf{C}_n/\partial_{n+1}$. $\square$

We now see that the weak n-replacement functor preserves trivial fibrations of crossed complexes.

Proposition 4.5.10. Let $f : [\mathsf{C}_1 : \mathsf{C} : \partial] \to [\mathsf{D}_1 : \mathsf{D} : \delta]$ be a trivial fibration of crossed complexes. Then there exists a trivial fibration $r^n(f) : r^n(\mathsf{C}) \to r^n(\mathsf{D})$.

Proof. For each $x \in \mathsf{C}_0$, the solid square

$$\begin{tikzcd}
\mathsf{C}(x)_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\partial_{n+1}"] & \mathsf{D}(f(x))_{n+1} \arrow[d, "\delta_{n+1}"] \\
\mathsf{C}(x)_n \arrow[r, "f_n"] \arrow[d] & \mathsf{D}(f(x))_n \arrow[d] \\
\mathsf{C}(x)_n/\mathrm{Im}\partial_{n+1} \arrow[r, dashed, "\overline{f}_n"'] & \mathsf{D}(f(x))_n/\mathrm{Im}\delta_{n+1}
\end{tikzcd}$$

commutes, so by the universal property of cokerδ_{n+1}, we have the unique morphism $\overline{f}_n$ which makes the above diagram commute for each $x \in \mathsf{C}_0$. Thus, we have a morphism of crossed complexes $r^n(f) : r^n(\mathsf{C}) \to r^n(\mathsf{D})$ defined by

$$r^n(f)_k = \begin{cases} f_k, & \text{if } k < n \\ \overline{f}_n, & \text{if } k = n \\ 0 & \text{if } k > n \end{cases}$$

Since f_n is a surjection of groups, $\overline{f}_n$ is a surjection. The morphism f_k was assumed to be surjective for $1 \le k < n$ and, clearly, the zero morphism is surjective for degrees geater than n. Thus, $r^n(f)$ is a fibration.

For every object $x \in \mathsf{C}_0$ and for all $1 \le k < n$, $r^n(\mathsf{C})(x)_k = \mathsf{C}(x)_k$ and $r^n(\mathsf{D})(x)_k = \mathsf{D}(x)_k$. Thus, $\pi_k(r^n(\mathsf{C}), x) = \pi_k(\mathsf{C}, x) \cong \pi_k(\mathsf{D}, f(x)) = \pi_k(r^n(\mathsf{D}), f(x))$ for all $x \in \mathsf{C}_0$ and $1 \le k < n-1$. Moreover, the homotopy groups for degrees greater than n of $r^n(\mathsf{C})$ and $r^n(\mathsf{D})$ are trivial. So we only need to check the homotopy groups for degrees $n-1$ and n. From the diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathsf{C}(x)_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d] & \mathsf{C}(x)_n \arrow[r, "\partial_n"] \arrow[d] & \mathsf{C}(x)_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, equal] & \mathsf{C}(x)_{n-2} \arrow[r] \arrow[d, equal] & \dots \\
\dots \arrow[r] & 0 \arrow[r, "\overline{\partial}_{n+1}"] & \mathsf{C}(x)_n/Im\partial_{n+1} \arrow[r, "\overline{\partial}_n"] & \mathsf{C}(x)_{n-1} \arrow[r, "\partial_{n-1}"] & \mathsf{C}(x)_{n-2} \arrow[r] & \dots
\end{tikzcd} \tag{4.9}$$

we see that for degree n,

$$\begin{aligned}\pi_{n-1}(r^n(\mathsf{C}), x) &= H(r^n(\mathsf{C})(x))_{n-1} \\ &= ker\overline{\partial}_{n-1}/Im\overline{\partial}_n \\ &= ker\partial_{n-1}/Im\partial_n \\ &= H(\mathsf{C}(x))_{n-1} \\ &= \pi_{n-1}(\mathsf{C}, x)\end{aligned}$$

Similarly, $\pi_{n-1}(r^n(\mathsf{D}), f(x)) = \pi_{n-1}(\mathsf{D}, f(x))$. Since f is trivial,

$$\pi_{n-1}(\mathsf{C}, x) \cong \pi_{n-1}(\mathsf{D}, f(x)).$$

Thus, we have the desired isomorphism of homotopy groups in degree $n-1$.

For degree n, we see from the diagram 4.9 that

$$\pi_n(r^n(\mathsf{C}), x) = H(r^n(\mathsf{C})(x))_n = ker\overline{\partial}_n/Im\overline{\partial}_{n+1} = ker\overline{\partial}_n \tag{4.10}$$

since $Im\overline{\partial}_{n+1} = 0$. Since ∂_n factors through $\overline{\partial}_n$,

$$ker\overline{\partial}_n \cong ker\partial_n/Im\partial_{n+1} = H(\mathsf{C}(x)) = \pi_n(\mathsf{C}, x) \tag{4.11}$$

Combining 4.10 and 4.11, we have that $\pi_n(r^n(\mathsf{C}), x) \cong \pi_n(\mathsf{C}, x)$. Similarly,

$$\pi_n(r^n(\mathsf{D}), f(x)) \cong \pi_n(\mathsf{D}, f(x)).$$

Since f is trivial,

$$\pi_n(r^n(\mathsf{C}), x) \cong \pi_n(r^n(\mathsf{D}), f(x)).$$

Lastly, note that taking a replacement does not affect the connected components so $\pi_0([r^n(\mathsf{C})]) \cong \pi_0([r^n(\mathsf{D})])$. Hence, there exists a trivial fibration $r^n(f) : r^n(\mathsf{C}) \to r^n(\mathsf{D})$. □

Corollary 4.5.11. Let $f : [\mathsf{C}_1 : \mathsf{C} : \partial] \to [\mathsf{D}_1 : \mathsf{D} : \delta]$ be a trivial fibration of crossed complexes where D is a n-crossed complex viewed as a crossed complex. Then there exists a trivial fibration $r^n(f) : r^n(\mathsf{C}) \to \mathsf{D}$.

We now see how the weak n-replacement functor acts on cofibrant objects and, furthermore, cofibrant replacements.

Proposition 4.5.12. Let $[\mathsf{Q}_1 : \mathsf{Q} : \xi]$ be a cofibrant object in $\mathbf{Xc}$. Then $r^n(\mathsf{Q})$ is a cofibrant object in $n\mathbf{Xc}$ viewed as a subcategory of $\mathbf{Xc}$.

Proof. Consider a trivial fibration $f : \mathsf{C} \to \mathsf{D}$ of n-crossed complexes. Then there is a lift l of the solid square

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathsf{C} \arrow[d] \\
\mathsf{Q} \arrow[r] \arrow[ur, dashed, "l"] & \mathsf{D}
\end{tikzcd} \tag{4.12}$$

for any morphism $\mathsf{Q} \to \mathsf{D}$ since Q is cofibrant. To show that $r^n(\mathsf{Q})$ is cofibrant, we need to find a lift $\bar{l}$ of the solid square

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathsf{C} \arrow[d] \\
r^n(\mathsf{Q}) \arrow[r] \arrow[ur, dashed, "\bar{l}"] & \mathsf{D}
\end{tikzcd} \tag{4.13}$$

for any morphism $r^n(\mathsf{Q}) \to \mathsf{D}$.

Taking the standard composition $\mathsf{Q} \to r^n(\mathsf{Q}) \to \mathsf{D}$ and the lift to C guaranteed from 4.12, we have the solid diagram

$$\begin{tikzcd}
 & & \vdots \arrow[d] & \\
\vdots \arrow[d] & \vdots \arrow[d] & 0 \arrow[d] \arrow[dr] & \vdots \arrow[d] \\
\mathsf{Q}_{n+1} \arrow[r, dotted] \arrow[d, "\xi_{n+1}"'] \arrow[urr, bend left, "l_{n+1}"] & 0 \arrow[r, dotted] \arrow[d] \arrow[ur, dashed, "\bar{l}_{n+1}"] & \mathsf{C}_n \arrow[r, dotted] \arrow[d] \arrow[dr] & 0 \arrow[d] \\
\mathsf{Q}_n \arrow[r, dotted] \arrow[d, "\xi_n"'] \arrow[urr, bend left, "l_n"] & \mathsf{Q}_n/Im\partial_{n+1} \arrow[r, dotted] \arrow[d] \arrow[ur, dashed, "\bar{l}_n"] & \mathsf{C}_{n-1} \arrow[r, dotted] \arrow[d] \arrow[dr] & \mathsf{D}_n \arrow[d] \\
\mathsf{Q}_{n-1} \arrow[r, dotted] \arrow[d] \arrow[urr, bend left, "l_{n-1}"] & \mathsf{Q}_{n-1} \arrow[rr, dotted] \arrow[d] \arrow[ur, dashed, "\bar{l}_{n-1}"] & \vdots & \mathsf{D}_{n-1} \arrow[d] \\
\vdots & \vdots & & \vdots
\end{tikzcd}$$

For degrees strictly less than n, defining $\bar{l} = l$ suffices. For degrees strictly greater than n, there is only one such choice, specifically $\bar{l} = 0$. For degree n, the diagram above shows that the image of ξ_{n+1} in Q_n maps to the identity of C_n. Thus, l_n factors through $\mathsf{Q}_n/Im\partial_{n+1}$ which gives the factor $\bar{l}_n$. The morphism $\bar{l}$ is clearly a morphism of n-crossed complexes. Thus, $r^n(\mathsf{Q})$ lifts to C in $n\mathbf{Xc}$. Hence, $r^n(\mathsf{Q})$ is cofibrant. $\square$

Proposition 4.5.13. Let $[\mathsf{Q}_1 : \mathsf{Q} : \xi]$ be a cofibrant replacement of $[\mathsf{C}_1 : \mathsf{C} : \partial]$ in $\mathbf{Xc}$. Then $r^n(\mathsf{Q})$ is a cofibrant replacement of $r^n(\mathsf{C})$ in $n\mathbf{Xc}$ viewed as a subcategory of $\mathbf{Xc}$.

Proof. Let Q be a cofibrant replacement of C in $\mathbf{Xc}$. Then we have the factorization

$$\begin{tikzcd} * \arrow[rr, hook] \arrow[dr, "i"'] & & \mathsf{C} \\ & \mathsf{Q} \arrow[ur, two heads, "p", "\simeq"'] & \end{tikzcd} \tag{4.14}$$

We claim that $r^n(\mathsf{Q})$ is a cofibrant replacement of $r^n(\mathsf{C})$ in $n\mathbf{Xc}$. From corollary 4.5.11, there is a trivial fibration $\overline{p} : r^n(\mathsf{Q}) \to r^n(\mathsf{C})$. Moreover, from lemma 4.5.12, $r^n(\mathsf{Q})$ is cofibrant. Hence, $r^n(\mathsf{Q})$ is a cofibrant replacement of $r^n(\mathsf{C})$ which lives in $n\mathbf{Xc}$. □

Corollary 4.5.14. Let $[\mathsf{C}_1 : \mathsf{C} : \partial]$ be a n-crossed complex. Then there is a cofibrant replacement of C in $n\mathbf{Xc}$.

CHAPTER 5

N-BUTTERFLIES

The results of the prior chapter imply that we can study certain connected n-homotopy types by studying reduced n-crossed complexes. In particular, we can study the morphisms of these homotopy types by understanding the morphisms of the homotopy category of crossed complexes. Similarly to crossed modules, these morphisms are a quotient set of *weak morphisms* with respect to homtopy. Weak morphisms rely on computing cofibrant replacements much like the 2-case and are even more difficult to compute. To avoid these computations, we will follow a similar procedure for constructing butterflies of crossed modules to construct n-butterflies of reduced n-crossed complexes. We now beginning by appropriately defining the proper morphisms of crossed complexes which we would like to model.

5.1 Derived Mapping $(n-1)$-Groupoid

For strict ω-groupoids $\mathcal{H}, \mathcal{G}$ and their respective images H, G under the equivalence of Theorem 4.2.15, we have the bijection

$$[\mathcal{H}, \mathcal{G}]_{\omega\mathbf{Grpd}} \cong [\mathsf{H}, \mathsf{G}]_{\mathbf{Xc}}$$

Moreover, we know that

$$[\mathsf{H}, \mathsf{G}]_{\mathbf{Xc}} \cong \mathbf{Xc}(QR\mathsf{H}, QR\mathsf{G})/\simeq$$

where R and Q are the fibrant and cofibrant replacement functors, respectively. In fact, theorem A.3.9 allows us to simplify the right side of the above isomorphism to

$$\begin{aligned}[\mathsf{H}, \mathsf{G}]_{\mathbf{Xc}} &= \mathbf{Xc}(QR\mathsf{H}, QR\mathsf{G})/\simeq \\ &\cong \mathbf{Xc}(Q\mathsf{H}, R\mathsf{G})/\simeq \\ &\cong \mathbf{Xc}(Q\mathsf{H}, \mathsf{G})/\simeq\end{aligned}$$

where the second isomorphism follows from the fact that all crossed complexes are fibrant.

The morphisms of the homotopy category from the reduced crossed complex H to the reduced crossed complex G is a quotient set which hides information. In order to have a more in-depth understanding of these morphisms, we will now expose and organize the hidden data using category theory. First we introduce the objects of interest.

Definition 5.1.1. Let H and G be reduced crossed complexes. A **weak morphism** from H to G is a morphism in $\mathbf{Xc}(\mathsf{Q},\mathsf{G})$ where Q is a cofibrant replacement of H.

The set of weak morphisms from H to G is the *derived set*

$$Rhom_{\mathsf{Q}}(\mathsf{H},\mathsf{G}) = \mathbf{Xc}(\mathsf{Q},\mathsf{G})$$

where Q is a cofibrant replacement of H. These morphisms are just fractions of the form

$$\begin{tikzcd} & \mathsf{Q} \arrow[dl, two heads, "p"'] \arrow[dr, "f"] \arrow[d, phantom, "\simeq"] & \\ \mathsf{H} \arrow[rr, dotted, "w"'] & & \mathsf{G} \end{tikzcd} \tag{5.1}$$

where $\mathsf{Q} \xrightarrow[p]{\simeq} \mathsf{H}$ is a cofibrant replacement of H. Here the dotted morphism ω implies that a weak morphism from H to G should represent an actual arrow from H to G.

Now, we need to devise the correct morphisms between these objects; in particular, these morphisms should encode the homotopy theory of the model structure. Unfortunately, the homotopy equivalence relation is not an actual map between two crossed complexes. So to form a category we will would like a map that represents the homotopies. Andrew Tonks showed that a homotopy is equivalent to a 1-fold left homotopies (see 4.4.2). Moreover, weak morphisms with 1-fold left homotopies can be described by the groupoid of the internal crossed complex.

Definition 5.1.2. Let H and G be crossed complexes. The **derived mapping groupoid**, denoted by $\mathbf{Rhom}(\mathsf{H},\mathsf{G})_1$, is the groupoid $\mathbf{XC}(\mathsf{Q},\mathsf{G})_1$ of the internal crossed complex where Q is a cofibrant replacement of H.

In particular, we have the isomorphism

$$[\mathsf{H},\mathsf{G}]_{\mathbf{Xc}} \cong \pi_0(\mathbf{Rhom}(\mathsf{H},\mathsf{G})).$$

For $f, g \in \mathbf{Rhom}(\mathsf{H},\mathsf{G})_0$, we will view a 1-fold left homotopies as a diagram of the form

$$\begin{tikzcd} & \mathsf{Q} \arrow[dl, two heads, "p"', "\simeq"] \arrow[dr, "f"] \arrow[dd, "\ominus" description] & \\ \mathsf{H} & & \mathsf{G} \\ & \mathsf{Q} \arrow[ul, two heads, "p", "\simeq"'] \arrow[ur, "g"'] \arrow[ur, phantom, "\Leftarrow h"] & \end{tikzcd}$$

We will say that the diagram commutes up to the homotopy h.

To show that these categories are well-defined with respect to cofibrant replacements, we have the following result.

Proposition 5.1.3. For reduced n-crossed complexes H and G and cofibrant replacements Q, Q' of H, the categories $\mathbf{Rhom}_{\mathsf{Q}}(\mathsf{H}, \mathsf{G})$ and $\mathbf{Rhom}_{\mathsf{Q}'}(\mathsf{H}, \mathsf{G})$ are equivalent.

Proof. From the right lifting property of trivial fibrations, we have the lift

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathsf{Q} \arrow[d, two heads, "p"', "\simeq"] \\
\mathsf{Q}' \arrow[r, two heads, "p", "\simeq"'] \arrow[ur, dashed, "l"] & \mathsf{H}
\end{tikzcd}$$

By the two out of three axiom of model categories, l is a weak equivalence. The morphism l induces a set map

$$l^* : Rhom_{\mathsf{Q}}(\mathsf{H}, \mathsf{G}) \to Rhom_{\mathsf{Q}'}(\mathsf{H}, \mathsf{G})$$

The map l^* also induces the map on homotopies in the usual way. In particular, for a homotopy $h : f \Rightarrow g$, we have the homotopy $l^*(h) : l(f) \Rightarrow l(g)$ given by the diagram

$$\begin{tikzcd}
\mathsf{Q}' \arrow[dr, "l"] \arrow[dd] & & \\
& \mathsf{Q} \arrow[d] \arrow[dr, "f"] & \\
\mathsf{Q}' \otimes \mathsf{I} \arrow[r] \arrow[rr, dashed, bend left, "l^*(h)"] & \mathsf{Q} \otimes \mathsf{I} \arrow[r] & \mathsf{G} \\
& \mathsf{Q} \arrow[u] \arrow[ur, "g"'] & \\
\mathsf{Q}' \arrow[uu] \arrow[ur, "l"'] & &
\end{tikzcd}$$

In fact, we have defined a functor

$$l^* : \mathbf{Rhom}_{\mathsf{Q}}(\mathsf{H}, \mathsf{G}) \to \mathbf{Rhom}_{\mathsf{Q}'}(\mathsf{H}, \mathsf{G}).$$

By 4.4.9, l is a homotopy equivalence. So there exists a morphism $k : \mathsf{Q} \to \mathsf{Q}'$ and homotopies $l \circ k \simeq 1_{\mathsf{Q}}$ and $k \circ l \simeq 1_{\mathsf{Q}'}$. Similarly to l^*, we can define a functor

$$k^* : \mathbf{Rhom}_{\mathsf{Q}'}(\mathsf{H}, \mathsf{G}) \to \mathbf{Rhom}_{\mathsf{Q}}(\mathsf{H}, \mathsf{G}).$$

For $f \in \mathbf{Rhom}_{\mathsf{Q}}(\mathsf{H}, \mathsf{G})$, $k^* \circ l^*(f) = f \circ l \circ k$. Since $l \circ k \simeq 1_{\mathsf{Q}}$, we have that $k^* \circ l^*(f) \simeq f$. Similarly, for $g \in \mathbf{Rhom}_{\mathsf{Q}'}(\mathsf{H}, \mathsf{G})$, we have that $l^* \circ k^*(g) \simeq g$. Hence, $l*$ is an equivalence of categories. $\square$

Definition 5.1.4. Let H and G be crossed complexes. The **derived crossed complex**, denoted by $\mathbf{Rhom}(\mathsf{H}, \mathsf{G})$, is the internal crossed complex $\mathbf{XC}(\mathsf{Q}, \mathsf{G})$ where Q is a cofibrant replacement of H.

Proposition 5.1.5. Let H and G be reduced n-crossed complexes. The crossed complex $\mathbf{Rhom}(\mathsf{H}, \mathsf{G})$ is a $(n-1)$-crossed complex.

Proof. The k-fold left homotopies are k-degree maps for $k \geq n$. Since the groups G_k are trivial for $k \geq n$, the k-fold left homotopies are the trivial maps. Hence, $\mathbf{Rhom}(\mathsf{H}, \mathsf{G})(f)_k$ is the trivial group for all $f \in \mathbf{Rhom}(\mathsf{H}, \mathsf{G})_0$ and $k \geq n$. □

The definition and proposition above lead to the following definitions.

Definition 5.1.6. Let H and G be reduced crossed complexes. The image of $\mathbf{Rhom}(\mathsf{H}, \mathsf{G})$ under the equivalence in Theorem 4.2.15 defines a strict ω-groupoid called the *derived mapping ω-groupoid* and denoted by $\underline{\mathbf{Rhom}}(\mathsf{H}, \mathsf{G})$. When H and G are reduced, $\underline{\mathbf{Rhom}}(\mathsf{H}, \mathsf{G})$ defines a $(n-1)$-groupoid called the *derived mapping $(n-1)$-groupoid.*

Remark 5.1.7. For this paper, we are only going to consider reduced crossed complexes. It appears that the following theory should be able to be adapted to crossed complexes; however, we will postpone this excursion to a later paper.

Now we begin are analysis of weak morphisms. As cofibrant objects are not very manageable, we would like to construct a crossed complex E algebraically which corresponds to a derived morphism ω given by a fraction

$$\begin{tikzcd}
& \mathsf{Q} \arrow[d] \arrow[ddl, "p"', "\simeq"] \arrow[ddr, "f"] & \\
& \mathsf{E} \arrow[dl, "p^*", "\simeq"'] \arrow[dr, "f^*"'] & \\
\mathsf{H} \arrow[rr, dotted, "w"'] & & \mathsf{G}
\end{tikzcd} \tag{5.2}$$

In particular, E should be at least slightly easier to deal with then Q. Thus, giving a more manageable model of the weak morphisms of reduced crossed complexes.

Proposition 5.1.8. For two reduced crossed complexes $[\mathsf{H} : \partial]$ and $[\mathsf{G} : \delta]$, the **product crossed complex** $[\mathsf{H} \times \mathsf{G} : \partial \times \delta]$ defined by $(\mathsf{H} \times \mathsf{G})_n = \mathsf{H}_n \times \mathsf{G}_n$ with differentials being the usual product is a crossed complex. Moreover, $\mathsf{H} \times \mathsf{G}$ with degree wise projections $\pi_1 : \mathsf{H} \times \mathsf{G} \to \mathsf{H}$ and $\pi_2 : \mathsf{H} \times \mathsf{G} \to \mathsf{G}$ is universal up to isomorphism in **Crs**.

Proof. Clear. □

Definition 5.1.9. A reduced crossed complex G is **acyclic** if all homotopy groups are trivial.

Proposition 5.1.10. Let $[\mathsf{H} : \partial]$ and $[\mathsf{G} : \delta]$ be reduced crossed complexes. If G is acyclic, then the induced morphism $\pi_1 : \mathsf{H} \times \mathsf{G} \to \mathsf{H}$ is a trivial fibration. Similarly, if H is acyclic, then the induced morphism $\pi_2 : \mathsf{H} \times \mathsf{G} \to \mathsf{G}$ is a trivial fibration.

Proof. The differentials of the crossed complex $\mathsf{H} \times \mathsf{G}$ are the product maps. Thus, $\pi_1(\mathsf{H} \times \mathsf{G}) = \mathrm{coker}\partial \times \mathrm{coker}\delta = \pi_1(\mathsf{H}) \times \pi_1(\mathsf{G})$ and $\pi_k(\mathsf{H} \times \mathsf{G}) = H(\mathsf{H})_k \times H(\mathsf{G})_k = \pi_k(\mathsf{H})$ for $k \geq 2$. Since G is acyclic, the right side of these products are zero. The fact that $\pi_1 : \mathsf{H} \times \mathsf{G} \to \mathsf{H}$ is a fibration is clear. A similar argument can be applied to $\pi_2 : \mathsf{H} \times \mathsf{G} \to \mathsf{H}$ when H is acyclic. $\square$

Remark 5.1.11. From the proposition above, one should think of the product crossed complex $\mathsf{H} \times \mathsf{G}$ as 'smack' in the middle of H and G, at least up to weak equivalence.

From diagram 5.1, we have a morphism of crossed complexes $p \times f : \mathsf{Q} \to \mathsf{H} \times \mathsf{G}$ which we will denote by ∇^f. Moreover, there is a factorization

$$\begin{tikzcd}
 & \mathsf{Q} \arrow[dl, "p"', "\simeq"] \arrow[d, "\nabla^f"] \arrow[dr, "f"] & \\
\mathsf{H} & \mathsf{H} \times \mathsf{G} \arrow[l, dashed, "\pi_1"] \arrow[r, dashed, "\pi_2"'] & \mathsf{G}
\end{tikzcd}$$

By the two out of three property of weak equivalences, if either ∇^f or π_1 is a weak equivalence, the other is guaranteed to be a weak equivalence. Unfortunately, neither is necessarily a weak equivalence; otherwise, $\mathsf{H} \times \mathsf{G}$ would be a suspect for E. So we want something "closer" to H and Q. In fact, such a crossed complex E would need to factor ∇^f.

5.2 Pushout of Weak 2-Replacements

We apply Behrang Noohi's pushout construction to weak 2-replacements of reduced crossed complexes instead of just crossed modules. Although we know that the weak 2-replacement defines a functor, we take a moment to break down the resulting crossed module structure, the canonical morphism and the induced morphisms. This will help in the construction of the *n-pushout below a morphism* in the next section.

Proposition 5.2.1. Let $[\mathsf{G} : \delta]$ be a reduced crossed complex. Then the induced morphism $\overline{\delta}_2 : \mathsf{G}_2/\delta_3 \to \mathsf{G}_1$ is a crossed module.

Proof. We know that $\delta_2 : \mathsf{G}_2 \to \mathsf{G}_1$ is a crossed module and that we have the factorization

$$\begin{tikzcd} \mathsf{G}_2 \arrow[r, "q_{\delta_3}"] \arrow[d, "\delta_2"'] & \mathsf{G}_2/\delta_3 \arrow[d, "\overline{\delta}_2"] \\ \mathsf{G}_1 \arrow[r, equal, "1_{\mathsf{G}_1}"'] & \mathsf{G}_1 \end{tikzcd} \tag{5.3}$$

in the category of groups since G is a chain complex. For $a \in \mathsf{G}_1$ and $[x] \in \mathsf{G}_2/\delta_3$, define an action of G_1 on G_2/δ_3 by $[x]^a = [x^a]$.

Let $[x] = [y]$ in G_2/δ_3. Then $x = y\partial_3(z)$. Since G_1 acts on the group G_2, we have

$$x^a = (y\partial_3(z))^a = y^a\partial_3(z)^a$$

Since G is a crossed complex, ∂_3 commutes with the action of G_1. So

$$y^a\partial_3(z)^a = y^a\partial_3(z^a)$$

In other words, $[x^a] = [y^a]$ in G_2/δ_3. Thus, $[x]^a = [x^a] = [y^a] = [y]^a$. Hence, the action is well-defined on G_2/δ_3.

With the action of G_1 on G_2, we have

$$[x]^{ab} = \left[x^{ab}\right] = \left[(x^a)^b\right] = [(x^a)]^b = ([x]^a)^b$$

and

$$[x]^1 = \left[x^1\right] = [x]$$

so the action of G_1 on G_2/δ_3 defined above is indeed an action. Also, since the action G_1 on G_2 preserves the group structure on G_2, we have that

$$[xy]^a = [(xy)^a] = [x^a y^a] = [x^a]\,[y^a] = [x]^a[y]^a$$

So the action of G_1 on G_2/δ_3 preserves the group structure on G_2/δ_3. Using the crossed module structure of $\delta_2 : \mathsf{G}_2 \to \mathsf{G}_1$ and the commutative diagram 5.3, we have that

$$\overline{\delta}_2\left([x]^a\right) = \overline{\delta}_2\left([x^a]\right) = \delta_2\left(x^a\right) = a^{-1}\delta_2\left(x\right)a = a^{-1}\overline{\delta}_2\left([x]\right)a$$

and

$$[y]^{-1}[x][y] = [y^{-1}xy] = \left[x^{\delta_2(y)}\right] = [x]^{\delta_2(y)} = [x]^{\overline{\delta}_2([y])}$$

Thus, $\overline{\delta}_2$ satisfies the two properties CM1 and CM2 of crossed modules. Hence, $\overline{\delta}_2 : \mathsf{G}_2/\delta_3 \to \mathsf{G}_1$ is a crossed module. $\square$

Proposition 5.2.2. Let $[\mathsf{G} : \delta]$ be a reduced crossed complex. Then the commutative square

$$\begin{tikzcd} \mathsf{G}_2 \arrow[r, "q_{\delta_3}"] \arrow[d, "\delta_2"'] & \mathsf{G}_2/\delta_3 \arrow[d, "\overline{\delta}_2"] \\ \mathsf{G}_1 \arrow[r, equal, "1_{\mathsf{G}_1}"'] & \mathsf{G}_1 \end{tikzcd}$$

is a morphism of crossed modules.

Proof. Since G is a chain complex, the commutativity of the diagram in the category of groups follows. We only need to check that q_{δ_3} is 1_{G_1}-invariant. This is easily seen from

$$q_{\delta_3}(x^a) = [x^a] = [x]^a = q_{\delta_3}(x)^a = q_{\delta_3}(x)^{1(a)}$$

□

Proposition 5.2.3. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then there is a morphism of crossed modules

$$\begin{tikzcd} \mathsf{H}_2/\partial_3 \arrow[r, "\overline{q_{\delta_3} \circ f_2}"] \arrow[d, "\overline{\partial}_2"] & \mathsf{G}_2/\delta_3 \arrow[d, "\overline{\delta}_2"] \\ \mathsf{H}_1 \arrow[r, "f_1"] & \mathsf{G}_1 \end{tikzcd}$$

Proof. By Proposition 5.2.1, both of the crossed modules exist and the commutativity follows from the definitions of the lifts and the fact that f is a morphism of reduced crossed complexes. We only need to show that $\overline{q_{\delta_3} \circ f_2}$ is f_1-invariant. For $a \in \mathsf{H}_1$ and $[x] \in \mathsf{H}_2/\partial_3$, we have that

$$\begin{aligned} \overline{q_{\delta_3} \circ f_2}\,([x]^a) &= \overline{q_{\delta_3} \circ f_2}\,([x^a]) \\ &= q_{\delta_3} \circ f_2\,(x^a) \\ &= q_{\delta_3}\,(f_2\,(x^a)) \end{aligned}$$

Since f is a morphism of crossed complexes, f_2 is f_1-invariant. Moreover, by 5.2.2, q_{δ_3} is 1_{G_1}-invariant. Thus,

$$\begin{aligned} q_{\delta_3}\,(f_2\,(x^a)) &= q_{\delta_3}\left(f_2\,(x)^{f_1(a)}\right) \\ &= q_{\delta_3}\,(f_2(x))^{1_{\mathsf{G}_1}(f_1(a))} \\ &= q_{\delta_3}\,(f_2(x))^{f_1(a)} \\ &= (q_{\delta_3} \circ f_2(x))^{f_1(a)} \\ &= \left(\overline{q_{\delta_3} \circ f_2}([x])\right)^{f_1(a)} \end{aligned}$$

Hence, we have the desired morphism of crossed modules. □

Proposition 5.2.4. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then there is the factorization

$$\begin{tikzcd}
\mathsf{H}_2 \arrow[r, "q_{\delta_3} \circ f_2"] \arrow[d, "\partial_2"'] & \mathsf{G}_2/\delta_3 \arrow[r, equal, "1_{\mathsf{G}_2/\delta_3}"] \arrow[d, dashed, "\bar{i}_2"] & \mathsf{G}_2/\delta_3 \arrow[d, "\bar{\delta}_2"] \\
\mathsf{H}_1 \arrow[r, dashed, "\bar{i}_1"'] & \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3 \arrow[r, "\rho"] & \mathsf{G}_1
\end{tikzcd}$$

given by the generalized pushout construction where $\bar{i}_2 : \mathsf{G}_2/\delta_3 \to \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$ defines a crossed module.

Proof. Taking the composition of morphisms of crossed complexes

$$\begin{tikzcd}
\mathsf{H}_2 \arrow[r, "f_2"] \arrow[d, "\partial_2"'] & \mathsf{G}_2 \arrow[r, "q_{\delta_3}"] \arrow[d, "\delta_2"'] & \mathsf{G}_2/\delta_3 \arrow[d, "\bar{\delta}_2"] \\
\mathsf{H}_1 \arrow[r, "f_1"'] & \mathsf{G}_1 \arrow[r, equal, "1_{\mathsf{G}_1}"'] & \mathsf{G}_1
\end{tikzcd}$$

we have a morphism from $\partial_2 : \mathsf{H}_2 \to \mathsf{H}_1$ to $\bar{\delta}_2 : \mathsf{G}_2/\delta_3 \to \mathsf{G}_1$ of crossed modules. Thus, by Behrang Noohi's work, we can construct the desired factorization above by the generalized pushout where $\bar{i}_2$ is a crossed module. $\square$

Lemma 5.2.5. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then

$$\mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3 \cong \mathsf{H}_1 \ltimes^{\mathsf{H}_2/\partial_3} \mathsf{G}_2/\delta_3$$

Proof. Both of these generalized pushouts are defined on the set $\mathsf{H}_1 \ltimes \mathsf{G}_2/\delta_3$ so we need only check that the respective normal groups

$$N = \{(\partial_2(z)^{-1}, q_{\delta_3} \circ f_2(z)) | z \in \mathsf{H}_2\}$$

and

$$\overline{N} = \{(\overline{\partial}_2([z])^{-1}, \overline{q_{\delta_3} \circ f_2}([z])) | [z] \in \mathsf{H}_2/\partial_3\}$$

Recall that the defining morphisms of $\overline{N}$ are precisely the lifts

$$\begin{tikzcd}
\mathsf{H}_2 \arrow[rr, "\partial_2"] \arrow[dr, "q_{\partial_3}"'] & & \mathsf{H}_1 \\
& \mathsf{H}_2/\partial_3 \arrow[ur, dashed, "\overline{p}_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{H}_2 \arrow[r, "f_2"] \arrow[dr, "q_{\partial_3}"'] & \mathsf{G}_2 \arrow[r, "q_{\delta_3}"] & \mathsf{G}_2/\delta_3 \\
& \mathsf{H}_2/\partial_3 \arrow[ur, dashed, "\overline{q_{\delta_3} \circ f_2}"'] &
\end{tikzcd}$$

which are defined since f is a morphism of chain complexes. Using the commutativity of these diagrams, we see that for $z \in \mathsf{H}_2$

$$\begin{aligned}(\partial_2(z)^{-1}, q_{\delta_3} \circ f_2(z)) &= (\overline{\partial}_2(q_{\partial_3}(z))^{-1}, \overline{q_{\delta_3} \circ f_2}((q_{\partial_3}(z))) \\ &= (\overline{\partial}_2([z])^{-1}, \overline{q_{\delta_3} \circ f_2}([z]))\end{aligned}$$

Hence, we have the desired isomorphism. ☐

Proposition 5.2.6. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then for the morphism of crossed modules

$$\begin{tikzcd}
\mathsf{H}_2/\partial_3 \arrow[r, "\overline{f_2}"] \arrow[d, "\overline{\partial}_2"] & \mathsf{G}_2/\delta_3 \arrow[d, "\overline{\delta}_2"] \\
\mathsf{H}_1 \arrow[r, "\bar{i}_1"] & \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3
\end{tikzcd}$$

the induced morphism

$$\pi_k\left(\overline{\partial}_2 : \mathsf{H}_2/\partial_3 \to \mathsf{H}_1\right) \to \pi_k\left(\overline{\delta}_2 : \mathsf{G}_2/\delta_3 \to \mathsf{G}_1\right)$$

is a surjection for $k = 2$ and an isomorphism for $k = 1$.

Proof. Using the isomorphism of Lemma 5.2.5, the morphism of crossed modules of 5.2.3 has the form

$$\begin{tikzcd}
\mathsf{H}_2/\partial_3 \arrow[r, "\overline{f_2}"] \arrow[d, "\overline{\partial}_2"] & \mathsf{G}_2/\delta_3 \arrow[d, "\overline{\delta}_2"] \\
\mathsf{H}_1 \arrow[r, "\bar{i}_1"] & \mathsf{H}_1 \ltimes^{\mathsf{H}_2/\partial_3} \mathsf{G}_2/\delta_3
\end{tikzcd}$$

This is just Behrang Noohi's pushout construction of the morphism of crossed modules in Proposition 5.2.3. Then the conclusion of this proposition follows from a result of Behrang Noohi. ☐

Proposition 5.2.7. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then the solid diagram is commutative

$$\begin{tikzcd}
\mathsf{H}_2 \arrow[r, "f_2"] \arrow[dd, "\partial_2"'] \arrow[drr, bend right] & \mathsf{G}_2 \arrow[rr, equal, "1_{\mathsf{G}_2}"] \arrow[dr, dashed, "q_{\delta_3}"] \arrow[dd, dotted, "i_2"] & & \mathsf{G}_2 \arrow[dr, "q_{\delta_3}"] \arrow[dd, dotted, "\delta_2"] & \\
& & \mathsf{G}_2/\delta_3 \arrow[rr, equal, "1_{\mathsf{G}_2/\delta_3}"] \arrow[dd, "\bar{i}_2"] & & \mathsf{G}_2/\delta_3 \arrow[dd, "\bar{\delta}_2"] \\
\mathsf{H}_1 \arrow[r, dotted, "i_1"] \arrow[drr, bend right, "\bar{i}_1"'] & \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2 \arrow[rr, dotted, "\rho"] \arrow[dr, dashed, "\overline{1\times q_{\delta_3}}"] & & \mathsf{G}_1 \arrow[dr, equal, "1_{\mathsf{G}_1}"] & \\
& & \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3 \arrow[rr, "\bar{\rho}"] & & \mathsf{G}_1
\end{tikzcd}$$

and there exists the dash morphisms of crossed modules

Proof. The commutativity of the solid diagram follows from the fact that the factorization from Proposition 5.2.4 was constructed using the generalized pushout of the diagram

$$\begin{tikzcd} \mathsf{H}_2 \arrow[r, "f_2"] \arrow[d, "\partial_2"'] & \mathsf{G}_2 \arrow[r, "q_{\delta_3}"] \arrow[d, "\delta_2"'] & \mathsf{G}_2/\delta_3 \arrow[d, "\bar{\delta}_2"] \\ \mathsf{H}_1 \arrow[r, "f_1"'] & \mathsf{G}_1 \arrow[r, equal, "1_{\mathsf{G}_1}"'] & \mathsf{G}_1 \end{tikzcd}$$

So we only need to show that there is a morphism of crossed modules

$$\begin{tikzcd} \mathsf{G}_2 \arrow[r, dashed, "q_{\delta_3}"] \arrow[d, "i_2"] & \mathsf{G}_2/\delta_3 \arrow[d, "\bar{i}_2"] \\ \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2 \arrow[r, dashed, "\overline{1\times q_{\delta_3}}"'] & \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3 \end{tikzcd} \tag{5.4}$$

All the morphisms are known except $\overline{1 \times q_{\delta_3}}$ so lets formerly define it. We know we have the solid diagram

$$\begin{tikzcd} \mathsf{H}_1 \ltimes \mathsf{G}_2 \arrow[r, "1\times q_{\delta_3}"] \arrow[dr, "q_N"'] & \mathsf{H}_1 \ltimes \mathsf{G}_2/\delta_3 \arrow[r, "q_{\overline{N}}"] & \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3 \\ & \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2 \arrow[ur, dashed, "\overline{1\times q_{\delta_3}}"'] & \end{tikzcd}$$

Let $z \in \mathsf{H}_2$. Then

$$\begin{aligned} q_{\overline{N}} \circ 1 \times q_{\delta_3}\left(\left(\partial_2(z)^{-1}, f_2(z)\right)\right) &= q_{\overline{N}}\left(1 \times q_{\delta_3}\left(\left(\partial_2(z)^{-1}, f_2(z)\right)\right)\right) \\ &= q_{\overline{N}}\left(\left(\partial_2(z)^{-1}, q_{\delta_3}\left(f_2(z)\right)\right)\right) \\ &= q_{\overline{N}}\left(\left(\partial_2(z)^{-1}, q_{\delta_3} \circ f_2(z)\right)\right) \qquad = [1,[1]] \end{aligned}$$

This morphism clearly makes diagram 5.4 commute in the category of groups.

Now, we only need to show that q_{δ_3} is $\overline{1 \times q_{\delta_3}}$-invariant. For $[a,b] \in \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2$ and $x \in \mathsf{G}_2$

$$\begin{aligned} q_{\delta_3}(x)^{\overline{1\times q_{\delta_3}}([a,b])} &= q_{\delta_3}(x)^{[1(a),q_{\delta_3}(b)]} \\ &= [x]^{[a,[b]]} \\ &= [b]^{-1}[x]^{f_1(a)}[b] \\ &= [b^{-1}]\left[x^{f_1(a)}\right][b] \\ &= \left[b^{-1}x^{f_1(a)}b\right] \\ &= \left[x^{[a,b]}\right] \\ &= q_{\delta_3}\left(x^{[a,b]}\right) \end{aligned}$$

Hence, we have the desired commutative diagram along with the desired morphism of crossed modules. $\square$

Proposition 5.2.8. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then the morphism $\bar{i}_2 \circ q_{\delta_3} : \mathsf{G}_2 \to \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$ is a crossed module. Moreover, the commutative diagram

$$\begin{tikzcd}
\mathsf{H}_2 \arrow[r, "f_2"] \arrow[d, "\partial_2"'] & \mathsf{G}_2 \arrow[d, "\bar{i}_2 \circ q_{\delta_3}"] \\
\mathsf{H}_1 \arrow[r, "\bar{i}_1"'] & \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3
\end{tikzcd}$$

is a morphism of crossed modules.

Proof. Let $[a,[b]] \in \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$ and $x \in \mathsf{G}_2$. Then the action of $\mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$ on G_2 is defined by $x^{[a,[b]]} = b^{-1}x^{f_1(a)}b$ where b is any representative of the equivalence class $[b]$ in G_2/∂_3. To see that this is well-defined, suppose $[b] = [c]$ in G_2/δ_3. Then $b = c\delta_3(z)$ for some $z \in \mathsf{G}_3$. Thus,

$$\begin{aligned}
x^{[a,[b]]} &= b^{-1}x^{f_1(a)}b \\
&= (c\delta_3(z))^{-1}\, x^{f_1(a)}\, (c\delta_3(z)) \\
&= \left(\delta_3(z)^{-1}c^{-1}\right) x^{f_1(a)}\, (c\delta_3(z)) \\
&= \delta_3(z)^{-1}\left(c^{-1}x^{f_1(a)}c\right)\delta_3(z)
\end{aligned}$$

Since $\delta_2 : \mathsf{G}_2 \to \mathsf{G}_1$ is a crossed module, by property CM2 of crossed modules, we have

$$\begin{aligned}
\delta_3(z)^{-1}\left(c^{-1}x^{f_1(a)}c\right)\delta_3(z) &= \left(c^{-1}x^{f_1(a)}c\right)^{\delta_2(\delta_3(z))} \\
&= \left(c^{-1}x^{f_1(a)}c\right)^{1} \\
&= c^{-1}x^{f_1(a)}c \\
&= x^{[a,[c]]}
\end{aligned}$$

Also, to make sure that this action is defined on the quotient $\mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$, suppose $z \in \mathsf{H}_2$. Then we have

$$\begin{aligned}
x^{\left(\partial_2(z)^{-1},\, q_{\delta_3}\circ f_2(z)\right)} &= x^{\left(\partial_2(z)^{-1},[f_2(z)]\right)} \\
&= f_2(z)^{-1}x^{f_1\left(\partial_2(z)^{-1}\right)}f_2(z) \\
&= f_2(z)^{-1}x^{f_1\left(\partial_2(z^{-1})\right)}f_2(z) \\
&= f_2(z)^{-1}x^{\delta_2\left(f_2(z^{-1})\right)}f_2(z)
\end{aligned}$$

Again, by property CM2 of δ_2, we have

$$\begin{aligned} f_2(z)^{-1}x^{\delta_2\left(f_2(z^{-1})\right)}f_2(z) &= f_2(z)^{-1}\left(f_2(z^{-1})^{-1}xf_2(z^{-1})\right)f_2(z) \\ &= f_2(z)^{-1}\left(f_2(z)xf_2(z)^{-1}\right)f_2(z) \\ &= \left(f_2(z)^{-1}f_2(z)\right)x\left(f_2(z)^{-1}f_2(z)\right) \\ &= x \end{aligned}$$

Thus,

$$N = \{(\partial_2(z)^{-1}, q_{\delta_3}\circ f_2(z))|z\in \mathsf{H}_2\}$$

acts trivially on G_2 so the action is well-defined on the quotient. Also, for $[c,[d]] \in \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$,

$$\begin{aligned} x^{([a,[b]][c,[d]])} &= x^{\left[ac,[b]^{f_1(c)}[d]\right]} \\ &= x^{\left[ac,\left[b^{f_1(c)}d\right]\right]} \\ &= \left(b^{f_1(c)}d\right)^{-1}x^{f_1(ac)}\left(b^{f_1(c)}d\right) \\ &= \left(d^{-1}\left(b^{f_1(c)}\right)^{-1}\right)x^{f_1(ac)}\left(b^{f_1(c)}d\right) \\ &= d^{-1}\left(\left(b^{f_1(c)}\right)^{-1}x^{f_1(a)f_1(c)}b^{f_1(c)}\right)d \\ &= d^{-1}(b^{-1}x^{f_1(a)}b)^{f_1(c)}d \\ &= (b^{-1}x^{f_1(a)}b)^{[c,[d]]} \\ &= \left(x^{[a,[b]]}\right)^{[c,[d]]} \end{aligned}$$

so the action defined above is indeed an action. For $y \in \mathsf{G}_2$, since the usual action of H_1 on G_2 via f preserves the group stucture of G_2, we have that

$$\begin{aligned} (xy)^{[a,[b]]} &= b^{-1}(xy)^a b \\ &= b^{-1}(x^a y^a)b \\ &= b^{-1}\left(x^a\left(bb^{-1}\right)y^a\right)b \\ &= \left(b^{-1}x^a b\right)\left(b^{-1}y^a b\right) \\ &= x^{[a,[b]]}y^{[a,[b]]} \end{aligned}$$

so the action preserves the group structure of G_2.

To check that

$$\partial_2^{f_2} : \mathsf{G}_2 \to \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$$

is a crossed module, we need to check that $\partial_2^{f_2}$ satisfies properties CM1 and CM2.

For CM1, let $x \in \mathsf{G}_2$ and $[a,[b]] \in \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$. Then we have

$$\begin{aligned}
[a,[b]]^{-1}\,\bar{i}_2 \circ q_{\delta_3}(x)\,[a,[b]] &= [a,[b]]^{-1}\,\bar{i}_2([x])\,[a,[b]] \\
&= [a,[b]]^{-1}\,[1,[x]]\,[a,[b]] \\
&= \left[a^{-1}, \left([b]^{-1}\right)^{a^{-1}}\right][1,[x]]\,[a,[b]] \\
&= \left[a^{-1}, \left([b]^{-1}\right)^{a^{-1}}\right]\left[a, [x]^{f_1(a)}[b]\right] \\
&= \left[a^{-1}, \left([b]^{-1}\right)^{a^{-1}}\right]\left[a, \left[x^{f_1(a)}\right][b]\right] \\
&= \left[a^{-1}a, \left([b]^{-1}\right)^{a^{-1}a}\left[x^{f_1(a)}\right][b]\right] \\
&= \left[1, [b]^{-1}\left[x^{f_1(a)}\right][b]\right] \\
&= \left[1, \left[b^{-1}x^{f_1(a)}b\right]\right] \\
&= \overline{i_2}\left(\left[b^{-1}x^{f_1(a)}b\right]\right) \\
&= \overline{i_2} \circ q_{\delta_3}\left(b^{-1}x^{f_1(a)}b\right) \\
&= \overline{i_2} \circ q_{\delta_3}\left(x^{[a,[b]]}\right)
\end{aligned}$$

For CM2, let $x, y \in \mathsf{G}_2$. Then we have

$$\begin{aligned}
x^{\overline{i_2}\circ q_{\delta_3}(y)} &= x^{[1,[y]]} \\
&= y^{-1}x^{f_1(1)}y \\
&= y^{-1}x^{1}y \\
&= y^{-1}x^{1}y
\end{aligned}$$

Thus,

$$\partial_2^{f_2} : \mathsf{G}_2 \to \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$$

is a crossed module.

By Proposition 5.2.7, the desired square is commutative. We only need to show that f_2 is $\bar{i}_1$-invariant.

For $a \in \mathsf{H}_1$ and $x \in \mathsf{H}_2$, since f_2 is f_1-invariant, we have that

$$\begin{aligned}
f_2(x^a) &= f_2(x)^{f_1(a)} \\
&= f_2(x)^{[a,[1]]} \\
&= f_2(x)^{\bar{i}_2(a)}
\end{aligned}$$

Thus, $\left(f_2, \bar{i}_1\right)$ is a morphism of crossed complexes. □

5.3 n-Pushout of Crossed Complexes

We now follow Behrang Noohi's work on generalized semi-direct products to formulate a pushout of a reduced crossed complex along a morphism of groups.

Definition 5.3.1. Let L be a group and the solid diagram

$$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "d"'] & G \arrow[d, dashed, "q_N \circ i_2"] \\ K \arrow[r, dashed, "q_N \circ i_1"'] & K \times^H G \end{tikzcd}$$

be a diagram of L-modules (abelian). Then the **fibered coproduct** of K and G under H is the group $K \times^H G = K \times G/N$ where $N = \{(d(h)^{-1}, p(h))|h \in H\}$.
The induced morphisms are the normal inclusions into the product i_1, i_2 followed by the quotient map $q_N : K \times G \to K \times^H G$.

As the morphisms $q_N \circ i_1$ and $q_N \circ i_2$ are cumbersome we will simply refer to these as the inclusions into the fibered coproduct and denote them by i_1 and i_2, respectively.

Remark 5.3.2. As H, G, K are abelian groups, the above quotient is well-defined. When commutativity is lost this is not necessarily true; however, as we will see for crossed complexes, a little non-commutativity is manageable.

Proposition 5.3.3. Let L be a group with the diagram of L-modules (abelian) as in definition 5.3.1. Then the induced morphism $\ker d \to \ker q_N \circ i_2$ is a surjection and the induced morphism $\mathrm{coker} d \to \mathrm{coker} q_N \circ i_2$ is an isomorphism.

Proof. By the universal properties of kernel and cokernel, we have the diagram

$$\begin{tikzcd} \ker d \arrow[r, "\overline{p}"] \arrow[d] & \ker q_N \circ i_2 \arrow[d] \\ H \arrow[r, "p"] \arrow[d, "d"'] & G \arrow[d, "q_N \circ i_2"] \\ K \arrow[r, "q_N \circ i_1"'] \arrow[d, "q_d"'] & K \times^H G \arrow[d] \\ \mathrm{coker} d \arrow[r] & \mathrm{coker} q_N \circ i_2 \end{tikzcd}$$

We first prove that $\overline{p}$ is surjective. Suppose $x \in \ker q_N \circ i_2$. Then we have that

$$[1, 1] = q_N \circ i_2(x) = [1, x]$$

So $(1,x) = (d(z)^{-1}, p(z))$ for some $z \in \mathsf{H}$. In other words, $d(z) = \left(d(z)^{-1}\right)^{-1} = 1^{-1} = 1$ and $p(z) = x$. Thus, $z \in \ker d$ and $\overline{p}(z) = p(z) = x$. Hence, $\overline{p}$ is surjective.

Now we prove that the morphism of cokernels is an isomorphism. Observe, the solid diagram

$$\begin{tikzcd} G \arrow[r, "i_2"] \arrow[dr, "q_N \circ i_2"'] & K \times G \arrow[r, "\pi_1"] \arrow[d, "q_N"] & K \arrow[r, "q_d"] & \operatorname{coker} d \\ & K \times^H G \arrow[urr, dashed, "\overline{q_d \circ \pi_1}"'] & & \end{tikzcd}$$

and let $z \in H$. Then

$$q_d \circ \pi_1 \left((d(z)^{-1}, p(z)\right) = q_d \left(\pi_1 \left((d(z)^{-1}, p(z)\right)\right) = q_d \left(d(z)^{-1}\right) = q_d \left(d(z^{-1})\right) = 1$$

Thus, the morphism $\overline{q_d \circ \pi_1}$ is well-defined.

Let $x \in G$. Then we have that

$$\begin{aligned} (\overline{q_d \circ \pi_1}) \circ (q_N \circ i_2)(x) &= \overline{q_d \circ \pi_1} \left(q_N \left(i_2(x)\right)\right) \\ &= q_d \left(\pi_1 \left(i_2(x)\right)\right) \\ &= q_d \left(\pi_1(1,x)\right) \\ &= q_d(1) \\ &= 1 \end{aligned}$$

So the map $(\overline{q_d \circ \pi_1}) \circ (q_N \circ i_2) : G \to K \ltimes^H G \to \operatorname{coker} d$ is the trivial map. Thus, by the universal property of the cokernel of $q_N \circ i_2$, there is a map $\operatorname{coker} q_N \circ i_2 \to \operatorname{coker} d$. Moreover, this must be an isomorphism. $\square$

Remark 5.3.4. In the proposition above, commutativity of the groups was assumed; however, since the proof did not require an abelian condition, the proposition holds for any case in which the fibered coproduct exists as constructed in definition 5.3.1.

Proposition 5.3.5. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then $N_k = \left\{(\partial_{k+1}(h)^{-1}, f_{k+1}(h)) | h \in \mathsf{H}_{k+1}\right\}$ is a normal H_1-submodule of $\mathsf{H}_k \times \mathsf{G}_{k+1}$ for $k \geq 2$ and N_1 is a normal H_1-submodule of $\mathsf{H}_1 \ltimes \mathsf{G}_2$. In fact, N_2 is in the center of $\mathsf{H}_2 \times \mathsf{G}_3$.

Proof. For all $k \geq 1$, we have the diagram of group homomorphisms

$$\begin{tikzcd} \mathsf{H}_{k+1} \arrow[r, "f_{k+1}"] \arrow[d, "\partial_{n+1}"'] & \mathsf{G}_{k+1} \\ \mathsf{H}_k & \end{tikzcd} \tag{5.5}$$

which are both group homomorphisms invariant under the action of H_1, so N_k is a H_1-submodule. For $k \geq 3$, the groups of diagram 5.5 are abelian, so N_k is clearly normal in $\mathsf{H}_k \times \mathsf{G}_{k+1}$ and from section 3.2 the normality of N_1 in $\mathsf{H}_1 \ltimes \mathsf{G}_2$ has been shown. We now show that N_2 is normal in $\mathsf{H}_2 \times \mathsf{G}_3$. For $h \in \mathsf{H}_2$ and $g \in \mathsf{G}_3$, we have

$$\begin{aligned}(1,g)^{-1}(\partial_3(h)^{-1}, f_3(h))(1,g) &= (1,g^{-1})(\partial_3(h)^{-1}, f_3(h))(1,g)\\ &= (\partial_3(h)^{-1}, g^{-1}f_3(h)g)\\ &= (\partial_3(h)^{-1}, f_3(h))\end{aligned}$$

The last equality follows from the fact that G_3 is abelian. Thus, viewing G_3 as a subgroup of $\mathsf{H}_2 \times \mathsf{G}_3$, G_3 centralizes N_2. For $a \in \mathsf{H}_2$ and $h \in \mathsf{H}_3$, property CM2 of crossed modules and the fact that ∂_3 preserves the trivial action of the image of H_2 on H_3 of crossed complexes, respectfully, guarantees

$$\begin{aligned}a^{-1}\partial_3(h)^{-1}a &= a^{-1}\partial_3(h)^{-1}a\\ &= (\partial_3(h)^{-1})^{\partial_2(a)}\\ &= \partial_3(h^{\partial_2(a)})^{-1}\\ &= \partial_3(h)^{-1}\end{aligned}$$

So for $(a,b) \in \mathsf{H}_2 \times \mathsf{G}_3$ and $h \in \mathsf{H}_3$, we have

$$\begin{aligned}(a,b)^{-1}(\partial_3(h)^{-1}, f_3(h))(a,b) &= (1,b)^{-1}(a,1)^{-1}(\partial_3(h)^{-1}, f_3(h))(a,1)(1,b)\\ &= (1,b)^{-1}(a,1)^{-1}(\partial_3(h)^{-1}, f_3(h))(a,1)(1,b)\\ &= (1,b)^{-1}(a^{-1}\partial_3(h)^{-1}a, f_3(h))(1,b)\\ &= (1,b)^{-1}(\partial_3(h)^{-1}, f_3(h))(1,b)\\ &= (\partial_3(h)^{-1}, f_3(h))\end{aligned}$$

where the last equality follows from the fact that G_3 centralizes N_2. Thus, N_2 is normal in $\mathsf{H}_2 \times \mathsf{G}_3$. In fact, N_2 is in the center of $\mathsf{H}_2 \times \mathsf{G}_3$. $\square$

Proposition 5.3.6. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then there is a lift of the morphism $\partial_k \circ \pi_1 : \mathsf{H}_k \times \mathsf{G}_{k+1}/\delta_{k+2} \to \mathsf{H}_k \to \mathsf{H}_{k-1}$ to

$$\overline{\partial_k \circ \pi_1} : \mathsf{H}_k \times^{\mathsf{H}_{k+1}} \mathsf{G}_{k+1}/\delta_{k+2} \to \mathsf{H}_{k-1}$$

for $k \geq 2$.

Proof. For $k \geq 2$, we have the solid diagram

$$\begin{tikzcd}
\mathsf{H}_k \times \mathsf{G}_{k+1}/\delta_{k+2} \arrow[r, "\pi_1"] \arrow[dr, "q_N"'] & \mathsf{H}_k \arrow[r, "\partial_k"] & \mathsf{H}_{k-1} \\
& \mathsf{H}_k \times^{\mathsf{H}_{k+1}} \mathsf{G}_{k+1}/\delta_{k+2} \arrow[ur, dashed, "\overline{\partial_k \circ \pi_1}"'] &
\end{tikzcd}$$

The morphism $\partial_k \circ \pi_1$ lifts to $\overline{\partial_k \circ \pi_1}$ since

$$\begin{aligned}
\partial_k \left(\pi_1 \left(\left(\partial_{k+1}(x)^{-1}, q_{\delta_{k+2}} \circ f_{k+1}(x)\right)\right)\right) &= \partial_k \left(\partial_{k+1}(x)^{-1}\right) \\
&= \partial_k \left(\partial_{k+1}(x^{-1})\right) \\
&= 1
\end{aligned}$$

for all $x \in \mathsf{H}_k$. $\square$

Definition 5.3.7. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then for $n \geq 1$ the n**-pushout** below f is the ordered pair $[\mathsf{H}^{f_n} : \partial^{f_n}]$ where $\mathsf{H}_k^{f_n}$ is defined by

$$\mathsf{H}_k^{f_n} = \begin{cases} \mathsf{G}_k \text{ for } k \geq n \\ \mathsf{H}_k \times^{\mathsf{H}_{k+1}} \mathsf{G}_{k+1}/\delta_{k+2} \text{ for } k = n-1 \quad (\ltimes \text{ when } n = 2) \\ \mathsf{H}_k \text{ for } k < n-1 \end{cases}$$

and $\partial_k^{f_n} : \mathsf{H}_k^{f_n} \to \mathsf{H}_{k-1}^{f_n}$ is defined by

$$\partial_k^{f_n} = \begin{cases} \delta_k \text{ for } k > n \\ \bar{i}_2 \circ q_{\delta_{n+1}} \text{ for } k = n \\ \overline{\partial_k \circ \pi_1} \text{ for } k = n-1 \\ \partial_k \text{ for } k < n-1 \end{cases}$$

Conjecture 5.3.8. Let $f : \mathsf{H} \to \mathsf{G}$ be a morphism of reduced crossed complexes. Then

$$\begin{aligned}
\mathsf{H}_{n-1} \times^{\mathsf{H}_n} \mathsf{G}_n/\delta_{n+1} &\cong \mathsf{H}_{n-1} \times^{\mathsf{H}_n/\partial_{n+1}} \mathsf{G}_n/\delta_{n+1} \\
&\cong \mathsf{H}_{n-1} \times^{H_n \times^{\mathsf{H}_{n+1}} \mathsf{G}_{n+1}} \mathsf{G}_n
\end{aligned}$$

for all $n \geq 2$.

Proof. The first isomorphism is merely a generalization of lemma 5.2.5. $\square$

Generally, the n-pushout below a morphism $f : \mathsf{H} \to \mathsf{G}$ has the form

$$\cdots \longrightarrow \mathsf{G}_{n+1} \longrightarrow \mathsf{G}_n \longrightarrow \mathsf{H}_{n-1} \times^{\mathsf{H}_n/} \mathsf{G}_n/\delta_{n+1} \longrightarrow \mathsf{H}_{n-2} \longrightarrow \cdots \longrightarrow \mathsf{H}_2 \longrightarrow \mathsf{H}_1$$

Proposition 5.3.9. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then $[\mathsf{H}^{f^n} : \partial^{f^n}]$ is a reduced crossed complex.

Proof. See appendix B.1. □

Proposition 5.3.10. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then $[\mathsf{H}^{f^n} : \partial^{f^n}]$ factors f.

Proof. See appendix B.2. □

Proposition 5.3.11. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then the induced morphism

$$\pi_k\left(r^n\left(\mathsf{H}\right)\right) \to \pi_k\left(r^n\left(\mathsf{H}^{f_n}\right)\right)$$

is a surjection for $k = n$ and an isomorphism for $k < n$.

Proof. In general, since

$$\mathsf{H}_{n-1} \times^{\mathsf{H}_n} \mathsf{G}_n/\delta_{n+1} \cong \mathsf{H}_{n-1} \times^{\mathsf{H}_n/\partial_{n+1}} \mathsf{G}_n/\delta_{n+1}$$

by proposition 5.3.3, the morphism

$$r^n\left(\mathsf{H}\right) \to r^n(\mathsf{H}^{f_n})$$

has the form

$$\begin{tikzcd}
\vdots \arrow[d] & & & \vdots \arrow[d] \\
\mathbf{0} \arrow[rrr] \arrow[d] & & & \mathbf{0} \arrow[d] \\
\mathsf{H}_n/\partial_{n+1} \arrow[rrr, "\overline{f}_n"] \arrow[d, "\overline{\partial}_n"'] & & & \mathsf{G}_n/\delta_{n+1} \arrow[d, "\overline{i}_2"] \\
\mathsf{H}_{n-1} \arrow[rrr, "\overline{i}_1"] \arrow[dr, "q_{\overline{\partial}_n}"] \arrow[dd, "\partial_{n-1}"'] & & & \mathsf{H}_{n-1} \times^{\mathsf{H}_n/\partial_{n+1}} \mathsf{G}_n/\delta_{n+1} \arrow[dl, "q_{\overline{i}_2}"'] \arrow[dd, "\overline{\partial_{n-1} \circ \pi_1}"] \\
& \mathsf{H}_{n-1}/\overline{\partial}_n \arrow[r, "\cong", "c"'] \arrow[dl, "\overline{\partial}_{n-1}"] & \left(\mathsf{H}_{n-1} \times^{\mathsf{H}_n/\partial_{n+1}} \mathsf{G}_n/\delta_{n+1}\right)/\overline{i}_2 \arrow[dr, "l"] & \\
\mathsf{H}_{n-2} \arrow[rrr, equal] \arrow[d] & & & \mathsf{H}_{n-2} \arrow[d] \\
\mathsf{H}_{n-3} \arrow[rrr, equal] \arrow[d] & & & \mathsf{H}_{n-3} \arrow[d] \\
\vdots & & & \vdots
\end{tikzcd}$$

Since the top nontrivial square is just the pushout, by proposition 5.3.8, the induced map

$$\ker \overline{\partial}_n = \pi_n \left(r^n \left(\mathsf{H}\right)\right) \to \pi_n \left(r^n \left(\mathsf{H}^{f_n}\right)\right) = \ker \bar{i}_2$$

is surjective.

Moreover,

$$\mathsf{H}_{n-1}/\overline{\partial}_n \cong \mathrm{coker}\overline{\partial_n} \cong \mathrm{coker}\bar{i}_2 \cong \left(\mathsf{H}_{n-1} \times^{\mathsf{H}_n/\partial_{n+1}} \mathsf{G}_n/\delta_{n+1}\right)/\bar{i}_2$$

Since this isomorphism of cokernels, denoted by c, is induced by $\bar{i}_1$, c satisfies the equation

$$c \circ \overline{\partial}_n = q_{\bar{i}_2} \circ \bar{i}_1$$

Since H^{f_n} is a chain complex, there is a lift

$$l : \left(\mathsf{H}_{n-1} \times^{\mathsf{H}_n/\partial_{n+1}} \mathsf{G}_n/\delta_{n+1}\right)/\bar{i}_2 \to \mathsf{H}_{n-2}$$

which satisfies

$$\begin{aligned}
l\left(c([x])\right) &= l\left(c\left(q_{\delta_{\overline{\partial}_n}}(x)\right)\right) \\
&= l\left(c\left(c^{-1}\left(q_{\bar{i}_2}\left(\bar{i}_1(x)\right)\right)\right)\right) \\
&= l\left(c \circ c^{-1}\left(q_{\bar{i}_2}\left(\bar{i}_1(x)\right)\right)\right) \\
&= l\left(q_{\bar{i}_2}\left(\bar{i}_1(x)\right)\right) \\
&= \overline{\partial_{n-1} \circ \pi_1}\left(\bar{i}_1(x)\right) \\
&= \partial_{n-1}(x) \\
&= \overline{\partial}_{n-1}\left(q_{\overline{\partial}_n}(x)\right) \\
&= \overline{\partial}_{n-1}(x)
\end{aligned}$$

So we have the commutative triangle

$$\begin{tikzcd}
\mathsf{H}_{n-1}/\overline{\partial}_n \arrow[rr, "\cong", "c"'] \arrow[dr, "\overline{\partial}_{n-1}"'] & & \left(\mathsf{H}_{n-1} \times^{\mathsf{H}_n/\partial_{n+1}} \mathsf{G}_n/\delta_{n+1}\right)/\bar{i}_2 \arrow[dl, "l"] \\
& \mathsf{H}_{n-2} &
\end{tikzcd}$$

Since c is an isomorphism, the induced morphism

$$\ker \overline{\partial}_{n-1} \to \ker l$$

is an isomorphism. Since

$$\ker \overline{\partial}_{n-1} \cong \ker \partial_{n-1}/\mathrm{Im}\partial_{n+1} = \pi_{n-1}\left(r^n\left(\mathsf{H}\right)\right)$$

and

$$\ker l \cong \ker \overline{\partial_{n-1} \circ \pi_1}/\mathrm{Im}\bar{i}_2 = \pi_{n-1}\left(r^n\left(\mathsf{H}^{f_n}\right)\right)$$

the induced morphism

$$\pi_{n-1}\left(r^n\left(\mathsf{H}\right)\right) \to \pi_{n-1}\left(r^n\left(\mathsf{H}^{f_n}\right)\right)$$

is an isomorphism.

For $k = n-2$, the kernels are in fact equal. So we will show that the images of ∂_{n-1} and $\overline{\partial_{n-1} \circ \pi_1}$ are equal. For $x \in \mathrm{Im}\partial_{n-1}$, there is a $z \in \mathsf{H}_{n-1}$ such that $\partial_{n-1}(z) = x$. Then by commutativity we have

$$\begin{aligned}\overline{\partial_{n-1} \circ \pi_1}\left([z,[1]]\right) &= \overline{\partial_{n-1} \circ \pi_1}\left(\bar{i}_1(z)\right)\\ &= \partial_{n-1}(z)\\ &= x\end{aligned}$$

Thus, $x \in \mathrm{Im}\overline{\partial_{n-1} \circ \pi_1}$.

Conversely, for $x \in \mathrm{Im}\overline{\partial_{n-1} \circ \pi_1}$, there exists an element $[u,[v]] \in \mathsf{H}_{n-1} \times^{\mathsf{H}_n/\partial_{n+1}} \mathsf{G}_n/\delta_{n+1}$ such that

$$\begin{aligned}x &= \overline{\partial_{n-1} \circ \pi_1}\left([u,[v]]\right)\\ &= l\left(q_{\bar{i}_2}[u,[v]]\right)\\ &= \overline{\partial}_{n-1}\left(c^{-1}\left(q_{\bar{i}_2}[u,[v]]\right)\right)\end{aligned}$$

Since $c^{-1}\left(q_{\bar{i}_2}[u,[v]]\right) \in \mathsf{H}_{n-1}/\overline{\partial}_n$ and $q_{\overline{\partial}_n}$ is surjective, there exists an element $z \in \mathsf{H}_{n-1}$ such that $q_{\overline{\partial}_n}(z) = c^{-1}\left(q_{\bar{i}_2}[u,[v]]\right)$. So we have that

$$\begin{aligned}\overline{\partial}_{n-1}\left(c^{-1}\left(q_{\bar{i}_2}[u,[v]]\right)\right) &= \overline{\partial}_{n-1}\left(q_{\overline{\partial}_n}(z)\right)\\ &= \partial_{n-1}(z)\end{aligned}$$

So $x \in \mathrm{Im}\partial_{n-1}$. Thus, the images are equal. Hence, we have the equality

$$\pi_{n-2}\left(r^n\left(\mathsf{H}\right)\right) = \pi_{n-2}\left(r^n\left(\mathsf{H}^{f_n}\right)\right)$$

For $k < n-2$, clearly we have the equality

$$\pi_k\left(r^n\left(\mathsf{H}\right)\right) = \pi_k\left(r^n\left(\mathsf{H}^{f_n}\right)\right)$$

by definition of H^{f_n}.

Hence, we have the desired results. $\square$

Corollary 5.3.12. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced n-crossed complexes. Then the induced morphism $\pi_k(\mathsf{H}) \to \pi_k(\mathsf{H}^{f_n})$ is a surjection for $k = n$ and an isomorphism for $1 \leq k < n$.

Proof. This follows from the fact that $r^n(\mathsf{H})_n = \mathsf{H}_n$ and $r^n(\mathsf{G}_n) = \mathsf{G}_n$ when H and G are reduced n-crossed complexes. $\square$

Proposition 5.3.13. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then the induced morphism $\pi_k(\mathsf{H}) \to \pi_k\left(\mathsf{H}^{f_n}\right)$ is a surjection for $k = n$ and an isomorphism $1 \leq k < n$. Moreover, $\pi_k\left(\mathsf{H}^{f_n}\right) \to \pi_k(\mathsf{G})$ is an isomorphism for $k > n$.

Proof. In general, the morphism f gives the factorization

$$\begin{tikzcd}
\vdots \arrow[d] & & & \vdots \arrow[d] & \vdots \arrow[d] \\
\mathsf{H}_{n+1} \arrow[rrr] \arrow[d, "\partial_{n+1}"'] & & & \mathsf{G}_{n+1} \arrow[r, equal] \arrow[d] & \mathsf{G}_{n+1} \arrow[d, "\delta_{n+1}"] \\
\mathsf{H}_n \arrow[rrr, "f_n"] \arrow[dr, "q_{\partial_{n+1}}"] \arrow[dd, "\partial_n"'] & & & \mathsf{G}_n \arrow[r, equal] \arrow[dl, "q_{\delta_{n+1}}"'] \arrow[dd, "\bar{i}_2 \circ q_{\delta_{n+1}}"] & \mathsf{G}_n \arrow[dd, "\delta_n"] \\
& \mathsf{H}_n/\partial_{n+1} \arrow[r, "\overline{f}_n"] \arrow[dl, "\overline{\partial}_n"'] & \mathsf{G}_n/\delta_{n+1} \arrow[dr, "\bar{i}_2"] & & \\
\mathsf{H}_{n-1} \arrow[rrr, "\bar{i}_1"] \arrow[d, "\partial_{n-1}"'] & & & \mathsf{H}_{n-1} \times^{\mathsf{H}_n/\partial_{n+1}} \mathsf{G}_n/\delta_{n+1} \arrow[r, equal] \arrow[d, "\overline{\partial_{n-1} \circ \pi_1}"] & \mathsf{G}_{n-1} \arrow[d, "\delta_{n-1}"] \\
\mathsf{H}_{n-2} \arrow[rrr, equal] \arrow[d] & & & \mathsf{H}_{n-2} \arrow[r, "f_{n-2}"'] \arrow[d] & \mathsf{G}_{n-2} \arrow[d, "\delta_{n-2}"] \\
\mathsf{H}_{n-3} \arrow[rrr, equal] \arrow[d] & & & \mathsf{H}_{n-3} \arrow[r, "f_{n-3}"'] \arrow[d] & \mathsf{G}_{n-3} \arrow[d, "\delta_{n-3}"] \\
\vdots & & & \vdots & \vdots
\end{tikzcd}$$

Clearly, we see that for $k \geq n + 1$

$$\pi_k\left(\mathsf{H}^{f_n}\right) = \pi_k(\mathsf{G})$$

and for $k \leq n - 2$

$$\pi_k\left(\mathsf{H}^{f_n}\right) = \pi_k(\mathsf{H})$$

by definition of $\mathsf{H}_k^{f_n}$.

Moreover, since $q_{\partial_{n+1}}$ is surjective and $\overline{\partial_n}$ factors ∂_n, the images of $\overline{\partial_n}$ and ∂_n are equal. Similarly, the images of $\bar{i}_2$ and $\bar{i}_2 \circ q_{\delta_{n+1}}$ are equal. Thus, by 5.3.11,

$$\pi_{n-1}(\mathsf{H}) \cong \pi_{n-1}\left(\mathsf{H}^{f_n}\right)$$

By 5.3.11, we have that the induced morphism

$$\ker \overline{\partial}_n = \pi_n(r^n(\mathsf{H})) \to \pi_n\left(r^n\left(\mathsf{H}^{f_n}\right)\right) = \ker \bar{i}_2$$

is a surjection. Since

$$\ker \overline{\partial}_n \cong \ker \partial_n / \mathrm{Im}\partial_{n+1} = \pi_n(\mathsf{H})$$

and

$$\ker \bar{i}_2 \cong \ker \bar{i}_2 \circ q_{\delta_{n+1}} / \mathrm{Im}\delta_{n+1} = \pi_n\left(\mathsf{H}^{f_n}\right)$$

the induced morphism

$$\pi_n(\mathsf{H}) \to \pi_n\left(\mathsf{H}^{f_n}\right)$$

is a surjection. □

5.4 n-Butterflies

We want to find an algebraic model of the weak morphisms between reduced crossed complexes.

Proposition 5.4.1. Let $f : [\mathsf{H}, \partial] \to [\mathsf{G}, \delta]$ be a morphism of reduced crossed complexes and Q a cofibrant replacentment of H. Then the factorization

$$r^n(\mathsf{Q}) \to r^n(\mathsf{Q}^{\nabla^f_n}) \to r^n(\mathsf{H} \times \mathsf{G})$$

induces the isomorphisms

$$\pi_k(r^n(\mathsf{Q})) \cong \pi_k\left(r^n(\mathsf{Q}^{\nabla^f_n})\right)$$

for all $1 \leq k \leq n$.

Proof. From proposition 5.3.11, we know the isomorphism holds for $1 \leq k < n$. Moreover, the induced morphism of homotopy groups

$$\pi_n(r^n(\mathsf{Q})) \to \pi_n\left(r^n(\mathsf{Q}^{\nabla_n})\right)$$

is a surjection. So we only need to show that it is injective.

The morphism $r^n(\mathsf{Q}) \to \left(\mathsf{Q}^{\nabla_n}\right)$ in degrees $n-1$ and n has the form

$$\begin{tikzcd}
\ker \overline{\xi}_n \arrow[r] \arrow[d] & \ker i_2 \arrow[d] \\
\mathsf{Q}_n/\xi_{n+1} \arrow[r, "\overline{q \circ \nabla}_n"] \arrow[d, "\overline{\xi}_n"'] & \mathsf{H}_n/\partial_{n+1} \times \mathsf{G}_n/\delta_{n+1} \arrow[d, "i_2"] \\
\mathsf{Q}_{n-1} \arrow[r, "i_1"] \arrow[d] & \mathsf{Q}_{n-1} \times^{\mathsf{Q}_n} \left(\mathsf{H}_n/\partial_{n+1} \times \mathsf{G}_n/\delta_{n+1}\right) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

The surjection of the kernels above is given explicitly by proposition 5.3.3.

Since $p : \mathsf{Q} \to \mathsf{H}$ is an acyclic fibration, we know that

$$r^n(\mathsf{Q}) \cong \ker \xi_{n-1} \times_{\ker \partial_{n-1}} \mathsf{H}_n/\partial_{n+1}$$

by proposition 4.5.6. Thus, diagram 5.4 has the form

$$\begin{tikzcd}
\ker \overline{\xi}_n \arrow[r] \arrow[d] & \ker i_2 \arrow[d] \\
\ker \xi_{n-1} \times_{\ker \partial_{n-1}} \mathsf{H}_n/\partial_{n+1} \arrow[r, "\overline{q \circ \nabla}_n"] \arrow[d, "\overline{\xi}_n"'] & \mathsf{H}_n/\partial_{n+1} \times \mathsf{G}_n/\delta_{n+1} \arrow[d, "i_2"] \\
\mathsf{Q}_{n-1} \arrow[r, "i_1"] \arrow[d] & \mathsf{Q}_{n-1} \times^{\mathsf{Q}_n} \left(\mathsf{H}_n/\partial_{n+1} \times \mathsf{G}_n/\delta_{n+1}\right) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

where $\overline{\xi}_n$ is the projection onto the first term and $\overline{p}_n$ is the projection on the second term. Let $(a, [b]) \in \ker \overline{\xi}_n$ such that $\overline{q \circ \nabla}_n \left((a, [b])\right) = ([1], [1])$. The first condition implies that

$$1 = \overline{\xi}_n \left(a, [b]\right) = a$$

The second implies that

$$\begin{aligned}
([1], [1]) &= \overline{q \circ \nabla}_n \left((a, [b])\right) \\
&= \left(q_{\partial_{n+1}} \circ \overline{p}_n \left((a, [b])\right), \overline{q_{\delta_{n+1}} \circ f_n} \left((a, [b])\right)\right) \\
&= \left([b], \overline{q_{\delta_{n+1}} \circ f_n} \left((a, [b])\right)\right)
\end{aligned}$$

So $a = 1$ and $[b] = [1]$. In other words, $(a, [b]) = (1, [1])$. Thus, the induced morphism

$$\pi_n \left(r^n(\mathsf{Q})\right) = \ker \overline{\xi}_n \to \ker i_2 = \pi_n \left(r^n(\mathsf{Q}^{\nabla_n})\right)$$

is injective, proving that the induced morphism of homotopy groups in degree n is in fact an isomorphism. $\square$

Corollary 5.4.2. Let $f : [\mathsf{H}, \partial] \to [\mathsf{G}, \delta]$ be a morphism of reduced n-crossed complexes and Q a cofibrant replacentment of H. Then the induced morphism $\mathsf{Q} \to \mathsf{Q}^{\nabla^f_n}$ is a weak equivalence.

Proof. This follows from the fact that $r^n(\mathsf{Q})_n = \mathsf{Q}_n$, $r^n(\mathsf{H})_n = \mathsf{H}_n$ and $r^n(\mathsf{G}_n) = \mathsf{G}_n$ when Q, H and G are reduced n-crossed complexes. $\square$

Corollary 5.4.3. Let $f : [\mathsf{H}, \partial] \to [\mathsf{G}, \delta]$ be a morphism of reduced n-crossed complexes and Q a cofibrant replacentment of H. Then the morphism

$$\mathsf{Q}^{\nabla^f_n} \xrightarrow{\rho} \mathsf{H} \times \mathsf{G} \xrightarrow{\pi_1} \mathsf{H}$$

is a trivial fibration.

Proof. Since the trivial fibration $\mathsf{Q} \overset{\simeq}{\underset{p}{\twoheadrightarrow}} \mathsf{H}$ factors as

$$\begin{tikzcd}
 & \mathsf{Q} \arrow[d, "\iota"] \arrow[ddl, two heads, "p", "\simeq"'] \\
 & \mathsf{Q}^{\nabla^f_n} \arrow[d, "\rho"] \\
 & \mathsf{H} \times \mathsf{G} \arrow[dl, "\pi_1"] \\
\mathsf{H} &
\end{tikzcd}$$

the morphism $\rho \circ \pi_1$ is a weak equivalence by 5.4.2 and the two out of three property of model categories. The fibration follows from the definition of fibration of reduced crossed complexes and the set theoretic fact about factorizations of set surjections. $\square$

Proposition 5.4.4. Let $f : [\mathsf{H}, \partial] \to [\mathsf{G}, \delta]$ be a morphism of reduced crossed complexes and Q a cofibrant replacentment of H. Then the induced chain complex

$$1 \to \mathsf{G}_n \xrightarrow{i} \mathsf{Q}^{\nabla^f_n}_{n-1} \xrightarrow{u} \ker \xi_{n-2} \times_{\ker \partial_{n-2}} \mathsf{H}_{n-1} \longrightarrow 1$$

is exact.

Proof. We first prove that i is injective. Suppose $a \in \mathsf{G}_n$ such that $i(a) = [1, (1, 1)]$. Then $[1, (1, a)] = [1, (1, 1)]$. So $(1, (1, a)) = \big(\xi_n(x)^{-1}, (p_n(x), f_n(x))\big)$ for some $x \in \mathsf{Q}_n$. Thus, $x \in \ker \xi_n$

and $p_n(x) = 1$. Since $p : \mathsf{Q} \to \mathsf{H}$ is a weak equivalence, p induces the isomorphism $\ker \xi_n \cong \ker \partial_n$. Thus, $x = 1$ and $a = f(x) = 1$. Hence, the sequence considered is exact at G_n.

Suppose $[a, (b, c)] \in \ker u$. Then

$$\begin{aligned}
(1, 1) &= u\left([a, (b, c)]\right) \\
&= (\xi_{n-1} \circ \pi_1([a, (b, c)]), \pi_1 \circ \rho_{n-1}([a, (b, c)])) \\
&= (\xi_{n-1}(a), \pi_1\left((p_{n-1}(a)\partial_n(b), f_{n-1}(a)\delta_n(x))\right)) \\
&= (\xi_{n-1}(a), p_{n-1}(a)\partial_n(b))
\end{aligned}$$

So $a \in \ker \xi_{n-1}$ and $p_{n-1}(a) = \partial_n(b^{-1})$. Moreover,

$$\rho_{n-1}\left([a, (1, 1)]\right) = \rho_{n-1} \circ \iota_{n-1}(a) = p_{n-1}(a) = \partial_n(b^{-1})$$

Since $a \in \ker \xi_{n-1}$, $[a, (1, 1)] \in \ker \eta^f_{n-1}$. Thus, when considering ρ as the induced morphism on homology $\pi_1 \circ \rho : H(\mathsf{Q}^{\nabla^f_n})_{n-1} \to H\left(\mathsf{H}\right)_{n-1}$, we have that

$$\pi_1 \circ \rho_{n-1}\left([[a, (1, 1)]]\right) = [\pi_1 \circ \rho_{n-1}\left([a, (1, 1)]\right)] = [1]$$

in $H\left(\mathsf{H}\right)_{n-1}$. Since ρ is a weak equivalence by 5.4.3, the induced morphism is in fact an isomorphism. So $[[a, (1, 1)]] = [[1, (1, 1)]]$ in $H\left(\mathsf{Q}^{\nabla^f_n}\right)_{n-1}$. Thus, there exists $(x, y) \in \mathsf{H}_n \times \mathsf{G}_n$ such that $[a, (1, 1)] = \eta^f_n\left((x, y)\right) = [1, (x, y)]$. So we have that

$$\begin{aligned}
[a, (b, c)] &= [a, (1, 1)][1, (b, c)] \\
&= [1, (x, y)][1, (b, c)] \\
&= [1, (xb, yc)]
\end{aligned}$$

Since

$$\partial_n(x) = \pi_1 \circ \rho_{n-1}\left([1, (x, y)]\right) = \pi_1 \circ \rho_{n-1}\left([a, (1, 1)]\right) = p_{n-1}(a)$$

we have that

$$\partial_n(xb) = \partial_n(x)\partial_n(b) = p_{n-1}(a)\partial_n(b) = 1$$

Since $xb \in \ker \partial_n$ and $\ker \xi_n = H(\mathsf{Q})_n \cong H(\mathsf{H})_n = \ker \partial_n$, there exists a $z \in \ker \xi_n$ such that $p_n(z) = xb$. So

$$\begin{aligned}
(1, (xb, yc)) &= \left(\xi_n(z)^{-1}, \left(p_n(z), ycf_n\left(z^{-1}\right) f_n(z)\right)\right) \\
&= \left(1, \left(1, ycf\left(z^{-1}\right)\right)\right) \left(\xi_n(z)^{-1}, (p_n(z), f_n(z))\right)
\end{aligned}$$

Thus, $[a,(b,c)] = [1,(xb,yc)] = \left[1,\left(1,ycf\left(z^{-1}\right)\right)\right]$. Hence, there exists a element of G_n, specifically $ycf\left(z^{-1}\right)$, which maps to $[a,(b,c)]$ in $\mathsf{Q}_{n-1}^{\nabla_n^f}$.

Since $p : \mathsf{Q} \to \mathsf{H}$ is a weak equivalence,

$$\mathsf{Q}_{n-1} \longrightarrow \ker \xi_{n-2} \times_{\ker \partial_{n-2}} \mathsf{H}_{n-1}$$

is a surjection. Thus, the complex

$$\mathsf{Q}_{n-1}^{\nabla_n^f} \longrightarrow \ker \xi_{n-2} \times_{\ker \partial_{n-2}} \mathsf{H}_{n-1} \longrightarrow 1$$

is exact.

Hence, the complete complex of the proposition is exact. $\square$

Definition 5.4.5. Let $[\mathsf{H},\partial], [\mathsf{G},\delta]$ be reduced n-crossed complexes and $[\mathsf{Q},\xi]$ be a cofibrant replacement of $[\mathsf{H},\partial]$. A n**-Butterfly** $([\mathsf{E},\eta],p,f,\alpha,\beta)$ from H to G is a commutative diagram

$$\begin{tikzcd}
\mathsf{H}_n \arrow[dd, "\partial_n"'] \arrow[dr, "\alpha"] & & \mathsf{G}_n \arrow[dl, "\beta"'] \arrow[dd, "\delta_n"] \\
& \mathsf{E}_{n-1} \arrow[dl, "p_n"'] \arrow[dr, "f_n"] \arrow[dd, "\eta_{n-1}"] & \\
\mathsf{H}_{n-1} \arrow[dd, "\partial_{n-1}"'] & & \mathsf{G}_{n-1} \arrow[dd, "\delta_{n-1}"] \\
& \mathsf{E}_{\leq n-2} \arrow[dl, "p"'] \arrow[dr, "f"] & \\
\mathsf{H}_{\leq n-2} & & \mathsf{G}_{\leq n-2}
\end{tikzcd}$$

where $\mathsf{E}_k = \mathsf{Q}_k$ for $k \leq n-2$,

$$[\mathsf{E} : \eta] \xrightarrow{p} [\mathsf{H}_{\leq n-1} : \partial] \qquad \text{and} \qquad [\mathsf{E} : \eta] \xrightarrow{f} [\mathsf{G}_{\leq n-1} : \partial]$$

are morphisms of reduced $(n-1)$-crossed complexes, the induced sequences

$$1 \longrightarrow \mathsf{G}_n \xrightarrow{\beta} \mathsf{E}_{n-1} \xrightarrow{u_n} \ker \eta_{n-2} \times_{\ker \partial_{n-2}} \mathsf{H}_{n-1} \longrightarrow 1$$

$$\mathsf{E}_k \xrightarrow{u_k} \ker \eta_{k-1} \times_{\ker \partial_{k-1}} \mathsf{H}_k \longrightarrow 1 \qquad \text{for } k \leq n-2$$

are exact, the compositions $\eta_{n-1} \circ (\alpha \times \beta)$ and $f_n \circ \alpha$ are complexes, and α, β satisfy the compatibility conditions

$$\alpha\left(x^{q_1(a)}\right) = \alpha(x)^a \qquad \text{and} \qquad \beta\left(y^{f_1(a)}\right) = \beta(y)^a$$

Definition 5.4.6. A morphism of n-butterflies from H to G

$$(\Theta, h) : ([\mathsf{E}, \eta], p, f, \alpha, \beta) \longrightarrow ([\mathsf{E}', \eta'], p', f', \alpha', \beta')$$

is given a morphism of reduced $(n-1)$-crossed complexes $\Theta : \mathsf{E} \to \mathsf{E}'$ such that Θ_{n-1} is a group isomorphism and $\Theta_k = 1_{\mathsf{E}_k}$ for $k \leq n-2$, and the diagram

$$\begin{tikzcd}
 & & & \mathsf{G}_n \arrow[dll] \arrow[ddl] \arrow[ddd] \\
\mathsf{H}_n \arrow[r] \arrow[drr] \arrow[ddd] & \mathsf{E}_{n-1} \arrow[dr, "\Theta_{n-1}"] \arrow[ddl, dotted] \arrow[dd, dotted] \arrow[ddrr, dotted] & & \\
 & & \mathsf{E}'_{n-1} \arrow[dll] \arrow[ddd] \arrow[dr] & \\
 & & & \mathsf{G}_{n-1} \arrow[ddd] \\
\mathsf{H}_{n-1} \arrow[ddd] & \mathsf{E}_{\leq n-2} \arrow[dr, equal] \arrow[ddl, dotted] \arrow[ddrr, dotted] & & \\
 & & \mathsf{E}'_{\leq n-2} \arrow[ddll] \arrow[dr] & \\
 & & & \mathsf{G}_{\leq n-2} \\
\mathsf{H}_{\leq n-2} & & &
\end{tikzcd}$$

commutes up to a homotopy

$$\begin{tikzcd}
\mathsf{E} \arrow[rr, "f"] \arrow[dr, "\Theta"'] & \arrow[d, phantom, "\Downarrow h"] & \mathsf{G} \\
 & \mathsf{E}' \arrow[ur, "f'"'] &
\end{tikzcd}$$

of morphisms between reduced $(n-1)$-crossed complexes.

Proposition 5.4.7. Let $[\mathsf{H}, \partial], [\mathsf{G}, \delta]$ be reduced n-crossed complexes. The n-butterflies from H to G along with the morphisms form a groupoid denoted by $\mathsf{B}^n(\mathsf{H}, \mathsf{G})$.

Proof. Clear. $\square$

Proposition 5.4.8. Let $[\mathsf{H}, \partial], [\mathsf{G}, \delta]$ be reduced n-crossed complexes and $([\mathsf{E}, \eta], p, f, \alpha, \beta)$ be a n-butterfly from H to G. Then the induced morphism

$$\begin{tikzcd} \mathsf{H}_n \times \mathsf{G}_n \arrow[r, "\pi_1"] \arrow[d, "\alpha \times \beta"'] & \mathsf{H}_n \arrow[d, "\partial_n"] \\ \mathsf{E}_{\leq n-1} \arrow[r, "p"'] & \mathsf{H}_{\leq n-1} \end{tikzcd}$$

of reduced n-crossed complexes is a trivial fibration.

Proof. By the exactness data of a butterfly and Proposition 4.5.6, we only need to prove that the induced morphism

$$\mathsf{H}_n \times \mathsf{G}_n \xrightarrow{k} \ker \eta_{n-1} \times_{\ker \partial_{n-1}} \mathsf{H}_n$$

is an isomorphism. Since the sequence

$$1 \to \mathsf{G}_n \xrightarrow{\beta} \mathsf{E}_{n-1} \xrightarrow{u} \ker \eta_{n-2} \times_{\ker \partial_{n-2}} \mathsf{H}_{n-1} \longrightarrow 1 \tag{5.6}$$

is an exact complex, the map u is a surjection.

Suppose $(a, b) \in \mathsf{H}_n \times \mathsf{G}_n$ such that $k((a, b)) = (1, 1)$. Then

$$(1, 1) = k((a, b)) = (\alpha(a)\beta(b), a)$$

Since $a = 1$ and $\beta(b) = \alpha(1)\beta(b) = \alpha(a)\beta(b) = 1$, $b \in ker\beta$. By the exactness of 5.6, β is injective. Furthermore, $b = 1$ so $(a, b) = (1, 1)$. Thus, k is injective.

Suppose $(a, b) \in \ker \eta_{n-1} \times_{\ker \partial_{n-1}} \mathsf{H}_n$. Then $\eta_{n-1}(a) = 1$ and $p_{n-1}(a) = \partial_n(b)$. Since

$$\partial_n(b) = \partial_n(\pi_1((b, 1))) = p_{n-1}(\alpha \times \beta(b, 1)) = p_{n-1}(\alpha(b)\beta(1)) = p_{n-1}(\alpha(b))$$

we have that

$$p_{n-1}(a\alpha(b)^{-1}) = p_{n-1}(a)p_{n-1}(\alpha(a))^{-1} = p_{n-1}(a)\partial_n(b)^{-1} = 1$$

So

$$\begin{aligned} u(a\alpha(b)^{-1}) &= \big(\eta_{n-1}(a\alpha(b)^{-1}), p_{n-1}(a\alpha(b)^{-1})\big) \\ &= \big(\eta_{n-1}(a)\eta_{n-1}(\alpha(b^{-1})), 1\big) \\ &= \big(\eta_{n-1}(\alpha \times \beta((b^{-1}, 1))), 1\big) \end{aligned}$$

Since $\mathsf{H}_n \times \mathsf{G}_n \to \mathsf{E}_n \to \mathsf{E}_{n-1}$ composes to zero, we have that

$$u(a\alpha(b)^{-1}) = (1, 1)$$

Thus, by exactness of 5.6, there exists an element $z \in \mathsf{G}_n$ such that $\beta(z) = a\alpha(b)^{-1}$. Thus,

$$\begin{aligned} k((b,z)) &= (\alpha \times \beta((b,z)), b) \\ &= (\alpha(b)\beta(z), b) \\ &= \big(\alpha(b)a\alpha(b)^{-1}, b\big) \\ &= \big(a\alpha(a)\alpha(b)^{-1}, b\big) \\ &= (a, b) \end{aligned}$$

Thus, u is surjective and indeed an isomorphism. Hence, the induced morphism is a trivial fibration by Proposition 4.5.6. □

Corollary 5.4.9. Let $[\mathsf{H}, \partial], [\mathsf{G}, \delta]$ be reduced n-crossed complexes and

$$(\Theta, h) : ([\mathsf{E}, \eta], p, f, \alpha, \beta) \longrightarrow ([\mathsf{E}', \eta'], p', f', \alpha', \beta')$$

be a morphism of n-butterflies. Then the induced morphism

$$\begin{array}{ccc} \mathsf{H}_n \times \mathsf{G}_n & = & \mathsf{H}_n \times \mathsf{G}_n \\ \downarrow & & \downarrow \\ \mathsf{E}_{\leq n-1} & \longrightarrow & \mathsf{E}'_{\leq n-1} \end{array}$$

of reduced n-crossed complexes is a weak equivalence.

Proof. By Proposition 5.4.8 and two out of three property of weak equivalences. □

5.5 Algebraic Model of Weak Morphisms

We now show that n-butterflies model weak morphisms by showing they model the connected components of the derived mapping groupoid. The last theorem of the section is the main result that there is a bijection

$$\pi_0 \mathbf{Rhom}(\mathsf{H}, \mathsf{G}) \cong \pi_0 \mathsf{B}^n(\mathsf{H}, \mathsf{G}).$$

Proposition 5.5.1. Let $[\mathsf{H} : \partial]$ and $[\mathsf{G} : \delta]$ be reduced n-crossed complexes. There is a set map

$$\Omega : \pi_0 \mathbf{Rhom}(\mathsf{H}, \mathsf{G}) \to \pi_0 \mathsf{B}^n(\mathsf{H}, \mathsf{G})$$

which sends a derived morphism to the induced n-butterfly by unfolding the n-pushout with respect to ∇.

Proof. We define a set map

$$\Omega : \pi_0 \mathbf{Rhom}(\mathsf{H}, \mathsf{G}) \longrightarrow \pi_0 \mathsf{B}^n(\mathsf{H}, \mathsf{G})$$

by assigning to a representative of a connected component of $\mathbf{Rhom}(\mathsf{H}, \mathsf{G})$ determined by $f : \mathsf{Q} \to \mathsf{G}$ the n-butterfly $\Omega(f)$ obtained by unfolding the map $\mathsf{Q}^{\nabla_n^f} \to \mathsf{H} \times \mathsf{G}$. Explicitly, we have the diagram

$$\begin{tikzcd}
\mathsf{H}_n \arrow[dr] \arrow[dd] & & \mathsf{G}_n \arrow[dl] \arrow[dd] \\
& \mathsf{Q}_{n-1} \times^{\nabla_n^f} (\mathsf{H}_n \times \mathsf{G}_n) \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathsf{H}_{n-1} \arrow[dd] & & \mathsf{G}_{n-1} \arrow[dd] \\
& \mathsf{Q}_{\leq n-2} \arrow[dl] \arrow[dr] & \\
\mathsf{H}_{\leq n-2} & & \mathsf{G}_{\leq n-2}
\end{tikzcd}$$

For simplicity, we will denote the middle $(n-1)$-reduced crossed complex by $[\mathsf{Q}^f : \xi^f]$.

We now check that this map is well-defined. Suppose $f, g : \mathsf{Q} \to \mathsf{G}$ determine representatives of the same connected component. Then there is a homotopy h from f to g. Explicitly,

$$\begin{tikzcd}
& \mathsf{Q} \arrow[dl, "p"'] \arrow[dr, "f"] \arrow[dd, "1_{\mathsf{Q}}"'] & \\
\mathsf{H} & \arrow[r, phantom, "\Downarrow h"] & \mathsf{G} \\
& \mathsf{Q} \arrow[ul, "p"] \arrow[ur, "g"'] &
\end{tikzcd} \tag{5.7}$$

Since $h : f \implies g$ is a homotopy, there is a pointed 1-fold left homotopy $(g, \phi_k : \mathsf{Q}_k \to \mathsf{G}_{k+1})$ such that

$$f_k(x) = g_k(x)\xi^g_{k+1}(\phi_k(x))\phi_{k-1}(\xi^f_k(x)) \qquad \text{for } 1 \leq k \leq n-1$$

where $\phi_0(*) = 1$ and

$$f_n(x) = g_n(x)\phi_{n-1}(\xi^f_n(x)).$$

We would like to define a morphism of n-butterflies

$$(\Theta, h^*) : \big([\mathsf{Q}^f, \xi^f], p, f, \alpha, \beta\big) \longrightarrow ([\mathsf{Q}^g, \xi^g], p, g, \alpha', \beta')$$

First, we define the isomorphism Θ_{n-1}. For the map $\iota_{n-1} : \mathsf{Q}_{n-1} \to \mathsf{Q}^f_{n-1}$, define

$$\iota^{\phi}_{n-1} : \mathsf{Q}_{n-1} \to \mathsf{Q}_{n-1} \times^{\nabla^g_n} (\mathsf{H}_n \times \mathsf{G}_n)$$

by $\iota^{\phi}_{n-1}(q) = [q, (1, \phi_{n-1}(q))]$. We claim that ι^{ϕ}_{n-1} makes the solid diagram

$$\begin{tikzcd}
 & \mathsf{H}_n \times \mathsf{G}_n \arrow[dr, "\xi^f_n"] \arrow[drr, bend left, "\xi^g_n"] & & \\
\mathsf{Q}_n \arrow[ur, "\nabla^f_n"] \arrow[dr, "\xi_n"'] & & \mathsf{Q}^f_{n-1} \arrow[r, dashed, "\Theta_{n-1}"] & \mathsf{Q}^g_{n-1} \\
 & \mathsf{Q}_{n-1} \arrow[ur, "\iota_{n-1}"] \arrow[urr, bend right, "\iota^{\phi}_{n-1}"'] & &
\end{tikzcd} \tag{5.8}$$

commute. We have that

$$\iota^{\phi}_{n-1}\left(\xi_n(q)\right) = [\xi_n(q)\,(1, \phi_{n-1}(\xi_n(q)))]$$

and

$$\begin{aligned}
\xi^g_n\left(\nabla^f_n(q)\right) &= \xi^g_n\left(p_n \times f_n(q)\right) \\
&= \xi^g_n\left((p_n(q), f_n(q))\right) \\
&= [1, (p_n(q), f_n(q))]
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&(1, (p_n(q), f_n(q)))\left(\xi_n\left(q\right), \left(1, \phi_{n-1}\left(\xi_n(q)\right)\right)\right)^{-1} \\
&= \left(\xi_n(q)^{-1}, \left(p_n(q), f_n(q)\phi_{n-1}\left(\xi_n(q)\right)^{-1}\right)\right) \\
&= \left(\xi_n(q)^{-1}, (p_n(q), g_n(q))\right) \\
&= \left(\xi_n(q)^{-1}, (p_n(q), g_n(q))\right)
\end{aligned}$$

since f is determined by the pointed 1-fold left homotopy $(g, \phi_k : \mathsf{Q}_k \to \mathsf{G}_{k+1})$. Thus,

$$\xi^f_n\left(\nabla^f_n(q)\right) = \iota^{\phi}_{n-1}\left(\xi_n(q)\right)$$

in the quotient $\mathsf{Q}_{n-1} \times^{\nabla^g_n} (\mathsf{H}_n \times \mathsf{G}_n)$.

By the universality of the fibered coproduct, there exists a unique morphism $\Theta_{n-1} : \mathsf{Q}^f_{n-1} \to \mathsf{Q}^g_{n-1}$ which makes the diagram 5.8 commute. Explicitly, Θ_{n-1} is defined by

$$\Theta_{n-1}([a, (b, c)]) = [a, (b, c\phi_{n-1}(a))]$$

Since $\mathsf{Q}^f_k = \mathsf{Q}^g_k = \mathsf{Q}_k$ for $1 \leq k \leq n-2$, one can easily check that setting $\Theta_k = 1_{\mathsf{Q}_k}$ for $1 \leq k \leq n-2$ then defines a morphism $\Theta : \mathsf{Q}^f \to \mathsf{Q}^g$ of reduced $(n-1)$-crossed complexes and the diagram of reduced $(n-1)$-crossed complexes

$$\begin{tikzcd}
 & \mathsf{Q}^f \arrow[dl, "\pi_1\circ\rho^f"'] \arrow[dd, "\Theta"] \\
\mathsf{H}_{<n} & \\
 & \mathsf{Q}^g \arrow[ul, "\pi_1\circ\rho^g"]
\end{tikzcd}$$

commutes.

Also since $\mathsf{Q}^f_k = \mathsf{Q}^g_k = \mathsf{Q}_k$ for $1 \leq k \leq n-2$, we let $\phi^*_k = \phi_k$ for $1 \leq k \leq n-2$. We need to check that ϕ^* respects the pointed 1-fold homotopy condition in degree $n-1$

$$\pi_2 \circ \rho^f\left([a,(b,c)]\right) = \left((\pi_2 \circ \rho^g) \circ \Theta_{n-1}\right)\left([a,(b,c)]\right)\phi_{n-2}\left(\xi^f_{n-1}([a,(b,c)])\right) \tag{5.9}$$

is satisfied so that the diagram of reduced $(n-1)$-crossed complexes

$$\begin{tikzcd}
 & \mathsf{Q}^f \arrow[dl, "\pi_1\circ\rho^f"'] \arrow[dd, "\Theta"] \arrow[dr, "\pi_2\circ\rho^f"] & \\
\mathsf{H}_{<n} & & \mathsf{G}_{<n} \arrow[dl, phantom, "\Swarrow h^*"] \\
 & \mathsf{Q}^g \arrow[ul, "\pi_1\circ\rho^g"] \arrow[ur, "\pi_2\circ\rho^g"'] &
\end{tikzcd}$$

commutes up to the homotopy h^* given by the 1-fold left homotopy ϕ^*. Now, we check that the homotopy condition in degree $n-1$. We know that

$$\begin{aligned}
\pi_2 \circ \rho^f\left([a,(b,c)]\right) &= \pi_2\left(\left(p_{n-1}(a), f_{n-1}(a)\right)\left(\partial_n(b), \delta_n(c)\right)\right) \\
&= \pi_2\left(\left(p_{n-1}(a)\partial_n(b), f_{n-1}(a)\delta_n(c)\right)\right) \\
&= f_{n-1}(a)\delta_n(c)
\end{aligned}$$

We now calculate the right-side of equation 5.9 the fact that f is determined by (g,ϕ) from the homotopy in 5.7.

For $n = 2$,

$$\begin{aligned}
&\left((\pi_2 \circ \rho^g) \circ \Theta_1\right)\left([a,(b,c)]\right)\phi_0\left(\xi^f_1([a,(b,c)])\right) \\
&= \pi_2 \circ \rho^g\left([a,(b,c\phi_1(a)]\right) \cdot 1 \\
&= \pi_2\left(\left(p_1(a), g_1(a)\right)\left(\partial_2(b), \delta_2\left(c\phi_1(a)\right)\right)\right) \\
&= \pi_2\left(\left(p_1(a)\partial_2(b), g_1(a)\delta_2\left(c\phi_1(a)\right)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= g_1(a)\delta_2\left(c\phi_1(a)\right) \\
&= g_1(a)\delta_2\left(\phi_1(a)\right)\delta_2(c) \\
&= f_1(a)\delta_2(c)
\end{aligned}
$$

For $n = 3$,

$$
\begin{aligned}
&\left((\pi_2 \circ \rho^g) \circ \Theta_2\right)([a,(b,c)])\phi_1\left(\xi_2^f([a,(b,c)])\right) \\
&= \pi_2 \circ \rho^g\left([a,(b,c\phi_2(a)]\right)\phi_1\left(\xi_2(a)\right) \\
&= \pi_2\left((p_2(a), g_2(a))\left(\partial_3(b), \delta_3\left(c\phi_2(a)\right)\right)\right)\phi_1\left(\xi_2(a)\right) \\
&= \pi_2\left((p_2(a)\partial_3(b), g_2(a)\delta_3\left(c\phi_2(a)\right))\right)\phi_1\left(\xi_2(a)\right) \\
&= g_2(a)\delta_3\left(c\phi_2(a)\right)\phi_1\left(\xi_2(a)\right) \\
&= g_2(a)\delta_3\left(\phi_2(a)c\right)\phi_1\left(\xi_2(a)\right) \\
&= g_2(a)\delta_3\left(\phi_2(a)\right)\delta_3(c)\phi_1\left(\xi_2(a)\right) \\
&= g_2(a)\delta_3\left(\phi_2(a)\right)\phi_1\left(\xi_2(a)\right)\phi_1\left(\xi_2(a)\right)^{-1}\delta_3(c)\phi_1\left(\xi_2(a)\right) \\
&= f_2(a)\phi_1\left(\xi_2(a)\right)^{-1}\delta_3(c)\phi_1\left(\xi_2(a)\right) \\
&= f_2(a)\delta_3(c)^{\delta_2((\phi_1(\xi_2(a)))} \\
&= f_2(a)\delta_3\left(c^{\delta_2((\phi_1(\xi_2(a)))}\right) \\
&= f_2(a)\delta_3\left(c\right)
\end{aligned}
$$

where the last equality follows from the definition of crossed complexes that δ_2 acts trivially.

For $n \geq 4$,

$$
\begin{aligned}
&\left((\pi_2 \circ \rho^g) \circ \Theta_{n-1}\right)([a,(b,c)])\phi_{n-2}\left(\xi_{n-1}^f([a,(b,c)])\right) \\
&= \pi_2 \circ \rho^g\left([a,(b,c\phi_{n-1}(a)]\right)\phi_{n-2}\left(\xi_{n-1}(a)\right) \\
&= \pi_2\left((p_{n-1}(a), g_{n-1}(a))\left(\partial_n(b), \delta_n\left(c\phi_{n-1}(a)\right)\right)\right)\phi_{n-2}\left(\xi_{n-1}(a)\right) \\
&= \pi_2\left((p_{n-1}(a)\partial_n(b), g_{n-1}(a)\delta_n\left(c\phi_{n-1}(a)\right))\right)\phi_{n-2}\left(\xi_{n-1}(a)\right) \\
&= g_{n-1}(a)\delta_n\left(c\phi_{n-1}(a)\right)\phi_{n-2}\left(\xi_{n-1}(a)\right) \\
&= g_{n-1}(a)\delta_n\left(\phi_{n-1}(a)\right)\phi_{n-2}\left(\xi_{n-1}(a)\right)\delta_n(c) \\
&= f_{n-1}(a)\delta_n(c)
\end{aligned}
$$

So we have the desired equality 5.9 for any n and, furthermore, a morphism of n-butterflies. Thus, $\Omega(f)$ and $\Omega(g)$ are in the same connected component of $\mathsf{B}^n(\mathsf{H},\mathsf{G})$. $\square$

Proposition 5.5.2. Let $[\mathsf{H} : \partial]$ and $[\mathsf{G} : \delta]$ be reduced n-crossed complexes. The set function

$$\Omega : \pi_0\mathbf{Rhom}(\mathsf{H}, \mathsf{G}) \to \pi_0\mathsf{B}^n(\mathsf{H}, \mathsf{G})$$

is an injection.

Proof. To show Ω is injective, suppose that we have two objects in $\mathbf{Rhom}(\mathsf{H}, \mathsf{G})$ determined by $f, g : \mathsf{Q} \to \mathsf{G}$ such that $\Omega(f)$ is in the same component as $\Omega(g)$. Then there is a morphism of n-butterflies

$$\begin{tikzcd}
 & & & \mathsf{G}_n \arrow[dll] \arrow[ddl] \arrow[ddd] \\
\mathsf{H}_n \arrow[r] \arrow[drr] \arrow[ddd] & \mathsf{Q}^f_{n-1} \arrow[dr, "\Theta_{n-1}"] \arrow[dddl, dotted] \arrow[ddd, dotted] \arrow[ddrr, dotted] & & \\
 & & \mathsf{Q}^g_{n-1} \arrow[ddll] \arrow[ddd] \arrow[dr] & \\
 & & & \mathsf{G}_{n-1} \arrow[ddd] \\
\mathsf{H}_{n-1} \arrow[ddd] & \mathsf{Q}^f_{\leq n-2} \arrow[dr, equal] \arrow[dddl, dotted] \arrow[ddrr, dotted] & & \\
 & & \mathsf{Q}^g_{\leq n-2} \arrow[ddll] \arrow[dr] & \\
 & & & \mathsf{G}_{\leq n-2} \\
\mathsf{H}_{\leq n-2} & & &
\end{tikzcd}$$

which commutes up to the homotopy

$$\begin{tikzcd}
\mathsf{Q}^f \arrow[rr, "\pi_2 \circ \rho^f"] \arrow[dr, "\Theta"'] \arrow[rr, phantom, "\Downarrow h", shift right=4ex] & & \mathsf{G} \\
 & \mathsf{Q}^g \arrow[ur, "\pi_2 \circ \rho^g"'] &
\end{tikzcd}$$

of reduced $(n-1)$-crossed complexes where Θ is an isomorphism in degree $n-1$ and the identity map in degrees $k \leq n-2$.

The homotopy h induces a pointed 1-fold left homotopy of reduced $(n-1)$-crossed complexes given by

$$(g \circ \Theta, \phi_k : \mathsf{Q}^f_k \to \mathsf{G}_{k+1}) \text{ for } 1 \leq k \leq n-1$$

Furthermore, f is determined by

$$f_k(x) = g_k(x)\delta_{k+1}(\phi_k(x))\phi_{k-1}(\xi_k(x)) \qquad\qquad \text{for } 1 \leq k < n-1$$

where $\phi_0(*) = 1$ and

$$(\pi_2 \circ \rho^f)_{n-1}([a,(b,c)]) = ((\pi_2 \circ \rho^g)_{n-1} \circ \Theta_{n-1})\left([a,(b,c)]\right)\phi_{n-2}(\xi^f_{n-1}([a,(b,c)])) \tag{5.10}$$

We want to show that f and g are in the same component. In other words, we want to find a homotopy h^* from f to g for a diagram

$$\begin{tikzcd}
 & \mathsf{Q} \arrow[dl, two heads, "p"', "\simeq"] \arrow[dr, "f"] \arrow[dd, "1_{\mathsf{Q}}"] \arrow[dd, phantom, shift left=6ex, "\Downarrow h^*"] & \\
\mathsf{H} & & \mathsf{G} \\
 & \mathsf{Q} \arrow[ul, two heads, "p", "\simeq"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

Equivalently, we may find a pointed 1-fold left homotopy

$$(g, \phi^*_k : \mathsf{Q}_k \to \mathsf{G}_{k+1}) \text{ for } 1 \leq k \leq n-1$$

such that

$$f_k(x) = g_k(x)\delta_{k+1}(\phi^*_k(x))\phi^*_{k-1}(\xi_k(x)) \qquad\qquad \text{for } 1 \leq k \leq n-1 \tag{5.11}$$

where $\phi_0(*) = 1$ and

$$f_n(x) = g_n(x)\phi^*_{n-1}(\xi_n(x)) \tag{5.12}$$

Since $\Theta_k : \mathsf{Q}^{\nabla^f_n} \to \mathsf{Q}^{\nabla^g_n}_k$ is the identity map for $k < n-1$, we let $\phi^*_k = \phi_k$ for $k < n-1$. Clearly we have that

$$f_k(x) = g_k(x)\delta_{k+1}(\phi^*_k(x))\phi^*_{k-1}(\xi_k(x))$$

for $k < n-1$ by definition of ϕ. So we only need to find an appropriate $\phi^*_{n-1} : \mathsf{Q}_{n-1} \to \mathsf{G}_n$ which satisfies the equations 5.11 and 5.12 for $k = n-1$ and $k = n$.

For the morphism $\nabla^f : \mathsf{Q} \to \mathsf{H} \times \mathsf{G}$ of reduced n-crossed complexes, we have the isomorphism

$$\mathsf{Q}^g_{n-1} = \mathsf{Q} \times^{\nabla^g_n} (\mathsf{H}_n \times \mathsf{G}_n) \cong \mathsf{Q}_n \times^{\overline{\nabla^g_n}} (\mathsf{H}_n \times \mathsf{G}_n / \nabla^g_n(\ker \xi_n))$$

Under this isomorphism, we have the morphisms of reduced n-crossed complexes in the solid diagram

$$\begin{tikzcd}
\mathsf{Q}_n \arrow[r] \arrow[d, "\xi_n"'] & \mathsf{H}_n \times \mathsf{G}_n/\nabla^g_n(\ker \xi_n) \arrow[d, "\overline{\xi^g_n}"] \\
\mathsf{Q}_{n-1} \arrow[r] \arrow[ur, dashed, "\varphi_{n-1}"] \arrow[d, "\xi_{n-1}"'] & \mathsf{Q}^g_{n-1} \arrow[d, "\xi^g_{n-1}"] \\
\mathsf{Q}_{\leq n-2} \arrow[r, equal] & \mathsf{Q}_{\leq n-2}
\end{tikzcd}$$

To define ϕ^*_{n-1}, we first define the map

$$\varphi_{n-1} : \mathsf{Q}_{n-1} \to \mathsf{H}_n \times \mathsf{G}_n/\nabla^g_n(\ker \xi_n).$$

First note that it is not necessary for φ_{n-1} to commute in the above diagram by the definition of a homotopy. Informally, for $q \in \mathsf{Q}_{n-1}$ we take $\varphi_{n-1}(q)$ to be the difference between $\Theta_{n-1}([q,(1,1)]_f)$ and $[q,(1,1)]_g$ in Q^g. More precisely, let $[a,(b,c)]_g = \Theta([q,(1,1)]_f)$ in $\mathsf{Q} \times^{\nabla^g_n} (\mathsf{H}_n \times \mathsf{G}_n)$. By proposition 5.3.3 and the universal property of cokernels on Θ, we have the commutative triangle

$$\begin{tikzcd}
 & \operatorname{coker}\xi_n \arrow[dl, "{[\iota^f_{n-1}]}"', "\cong"] \arrow[dr, "{[\iota^g_{n-1}]}", "\cong"'] & \\
\operatorname{coker}\xi^f_n \arrow[rr, "{[\Theta_n]}", "\cong"'] & & \operatorname{coker}\xi^g_n
\end{tikzcd}$$

So $[a] = [q]$ in $\operatorname{coker}\xi_n$. Thus, $a = q\xi_n(x)$ for some $x \in \mathsf{Q}_n$. Moreover, we have that

$$\begin{aligned}
\Theta([q,(1,1)]_f) &= [a,(b,c)]_g \\
&= [q\xi_n(x),(b,c)]_g \\
&= \left[q\xi_n(x),(b,c)\nabla^g_n(x)\nabla^g_n(x)^{-1}\right]_g \\
&= \left[q,(b,c)\nabla^g_n(x)]_g[\xi_n(x),\nabla^g_n(x)^{-1}\right]_g \\
&= \left[q,(b,c)\nabla^g_n(x)]_g[\xi_n(x^{-1})^{-1},\nabla^g_n(x^{-1})\right]_g \\
&= [q,(bp_n(x),cg_n(x))]_g \\
&= [q,(1,1)]_g\,[1,(bp_n(x),cg_n(x))]_g
\end{aligned}$$

We show that the map $\varphi_{n-1} : \mathsf{Q}_{n-1} \to \mathsf{H}_n \times \mathsf{G}_n/\nabla^f_n(\ker \xi_n)$ defined by $\varphi_{n-1}(q) = [bp_n(x),cg_n(x)]_g$ is well-defined. Suppose $(a',(b',c'))$ is another representative of $\Theta([q,(1,1)]_f)$ i.e. $\Theta([q,(1,1)]_f) = [a',(b',c')]$. Similarly to a, there exists $y \in \mathsf{Q}_n$ such that $a' = q\xi_n(y)$ and $\Theta([q,(1,1)]_f) = [q,(b'p_n(y),c'g_n(y))]_g$. So $\varphi_{n-1}(q) = [b'p_n(y),c'g_n(y)]_g$. Since

$$[q,(bp_n(x),cg_n(x))] = \Theta([q,(1,1)]) = \left[q,\left(b'p_n(y),c'g_n(y)\right)\right]$$

in Q^g_{n-1}, there exists a $z \in \mathsf{Q}_n$ such that

$$\begin{aligned}\left(\xi_n(z)^{-1}, \nabla^g_n(z)\right) &= \left(q, (bp_n(x), cg_n(x))\right)\left(q, \left(b'p_n(y), c'g_n(y)\right)\right)^{-1} \\ &= \left(1, (bp_n(x), cg_n(x))\left(b'p_n(y), c'g_n(y)\right)^{-1}\right)\end{aligned}$$

The equality implies that $(bp_n(x), cg_n(x))\,(b'p_n(y), c'g_n(y))^{-1} = \nabla^g_n(z)$ where $z \in \ker \xi_n$. So $[bp_n(x), cg_n(x)] = [b'p_n(y), c'g_n(y)]$ in $\mathsf{H}_n \times \mathsf{G}_n/\nabla^g_n(\ker \xi_n)$. Thus, the map $\mathsf{Q}_{n-1} \to \mathsf{H}_n \times \mathsf{G}_n/\nabla^g_n(\ker \xi_n)$ is well-defined. Moreover, this is clearly a group homomorphism.

We have the morphism $\mathsf{C} \to \mathsf{D}$ of reduced n-crossed complexes given by

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbf{1} \arrow[r] \arrow[d] & \ker \xi^g_n \arrow[r, "i"] \arrow[d] & \mathsf{H}_n \times \mathsf{G}_n \arrow[r] \arrow[d] & \mathbf{1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbf{1} \arrow[r] \arrow[d] & \mathsf{Q}_1 \arrow[d, equal] \\
\cdots \arrow[r] & \mathbf{1} \arrow[r] & \mathbf{1} \arrow[r] & \mathsf{H}_n \times \mathsf{G}_n/\nabla^g_n(\ker \xi_n) \arrow[r] & \mathbf{1} \arrow[r] & \cdots \arrow[r] & \mathbf{1} \arrow[r] & \mathsf{Q}_1
\end{tikzcd}$$

where $\mathsf{H}_n \times \mathsf{G}_n$ is in degree $n-1$. Since $\nabla^g_n(\ker \xi_n) \cong \ker \xi^g_n$, the map $\mathsf{C} \to \mathsf{D}$ is a trivial fibration. Since Q is cofibrant, there exists a lift

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathsf{C} \arrow[d, two heads, "\simeq"] \\
\mathsf{Q} \arrow[r] \arrow[ur, dashed, "\varphi^*_{n-1}"] & \mathsf{D}
\end{tikzcd}$$

where $\mathsf{Q} \to \mathsf{D}$ is the morphism of crossed complexes which is φ_{n-1} in degree $n-1$ and the trivial map in all other degrees. We define ϕ^*_{n-1} to be $\pi_2 \circ \varphi^*_{n-1} : \mathsf{Q}_{n-1} \to \mathsf{H}_n \times \mathsf{G}_n \to \mathsf{G}_n$. So we have the commutative solid diagram

$$\begin{tikzcd}
\mathsf{Q}_n \arrow[r, "\nabla^g_n"] \arrow[dd, "\xi_n"'] & \mathsf{H}_n \times \mathsf{G}_n \arrow[r, equal] \arrow[d, "q_\nabla"] & \mathsf{H}_n \times \mathsf{G}_n \arrow[r, "\pi_2"] \arrow[d, "q_\nabla"] & \mathsf{G}_n \arrow[dd, "\delta_n"] \\
 & \mathsf{H}_n \times \mathsf{G}_n/\nabla^g_n(\ker \xi_n) \arrow[r, equal] \arrow[d, "\overline{\xi^f_n}"] & \mathsf{H}_n \times \mathsf{G}_n/\nabla^g_n(\ker \xi_n) \arrow[d, "\overline{\partial \times \delta}"] & \\
\mathsf{Q}_{n-1} \arrow[r, "\iota_{n-1}"'] \arrow[uur, dotted, "\varphi^*_{n-1}"] \arrow[ur, dotted, "\varphi_{n-1}"'] & \mathsf{Q}_{n-1} \times^{\nabla^g_n} \mathsf{H}_n \times \mathsf{G}_n \arrow[r, "\rho^g"'] & \mathsf{H}_{n-1} \times \mathsf{G}_{n-1} \arrow[r, "\pi_2"'] & \mathsf{G}_{n-1}
\end{tikzcd} \tag{5.13}$$

where $\varphi_{n-1} = q_\nabla \circ \varphi^*_{n-1}$. Now we must show that (g, ϕ^*_k) satisfies the remaining cases of equations 5.11 and 5.12. Specifically, we need to check the case when $k = n-1$ and $k = n$.

For $k = n-1$,

$$\begin{aligned}&g_{n-1}(q)\delta_n(\phi^*_{n-1}(q))\phi^*_{n-2}(\xi_{n-1}(q)) \\ &= g_{n-1}(q)\delta_n(\phi^*_{n-1}(q))\phi_{n-2}(\xi_{n-1}(q)) \\ &= \pi_2 \circ \rho^g\left(\iota^g(q)\right)\delta_n(\phi^*_{n-1}(q))\phi_{n-2}(\xi_{n-1}(q))\end{aligned}$$

$$
\begin{aligned}
&= \pi_2 \circ \rho^g \left([q,(1,1)]_g\right) \delta_n(\phi^*_{n-1}(q))\phi_{n-2}(\xi_{n-1}(q)) \\
&= \pi_2 \circ \rho^g \left(\Theta_{n-1}\left([q,(1,1)]_f\right)[1,(bp_n(x),cg_n(x))]_g^{-1}\right) \delta_n(\phi^*_{n-1}(q))\phi_{n-2}(\xi_{n-1}(q)) \\
&= \pi_2 \circ \rho^g \left(\Theta_{n-1}\left([q,(1,1)]_f\right)\right) \pi_2 \circ \rho^g \left([1,(bp_n(x),cg_n(x))]_g^{-1}\right) \\
&\qquad \cdot \delta_n(\phi^*_{n-1}(q))\phi_{n-2}(\xi_{n-1}(q)) \\
&= \pi_2 \circ \rho^g \left(\Theta_{n-1}\left([q,(1,1)]_f\right)\right) \pi_2 \circ \rho^g \left([1,(bp_n(x),cg_n(x))]_g\right)^{-1} \\
&\qquad \cdot \delta_n(\phi^*_{n-1}(q))\phi_{n-2}(\xi_{n-1}(q)) \\
&= \pi_2 \circ \rho^g \left(\Theta_{n-1}\left([q,(1,1)]_f\right)\right) \delta_n(\phi^*_{n-1}(q))^{-1}\delta_n(\phi^*_{n-1}(q))\phi_{n-2}(\xi_{n-1}(q)) \\
&= \pi_2 \circ \rho^g \left(\Theta_{n-1}\left([q,(1,1)]_f\right)\right) \phi_{n-2}(\xi_{n-1}(q)) \\
&= \pi_2 \circ \rho^g \left(\Theta_{n-1}\left([q,(1,1)]_f\right)\right) \phi_{n-2}(\xi^f_{n-1}(q)) \\
&= f(q)
\end{aligned}
$$

where the equality $\pi_2 \circ \rho^g \left([1,(bp_n(x),cg_n(x))]_g\right) = \delta_n(\phi^*_{n-1}(q))$ follows from the diagram 5.13 and the last equality follows from the equation 5.10.

We now check the $k = n$ case. Since there is a morphism between the n-butterflies, we have the commutative diagram

$$
\begin{tikzcd}
 & \mathsf{H}_n \times \mathsf{G}_n \arrow[dl, "\xi^f_n"'] \arrow[dr, "\xi^g_n"] & \\
\mathsf{Q}^f_{n-1} \arrow[rr, "\Theta_{n-1}", "\cong"'] & & \mathsf{Q}^g_{n-1}
\end{tikzcd}
$$

$$
\begin{aligned}
[1,(p_n(q),f_n(q))]_g &= \xi^g_n(p_n(q)f_n(q)) \\
&= \Theta_{n-1}(\xi^f_n(p_n(q),f_n(q))) \\
&= \Theta_{n-1}([1,(p_n(q),f_n(q))]_f) \\
&= \Theta_{n-1}([\xi_n(q)^{-1},(1,1)]_f) \\
&= [\xi_n(q)^{-1},(1,1)]_g[1,\varphi_{n-1}(\xi_n(q))]_g \\
&= [1,(p_n(q),g_n(q))]_g[1,\varphi_{n-1}(\xi(q))]_g
\end{aligned}
$$

This equality implies that $\varphi_{n-1}(q) = [1, g_n(q)^{-1}f_n(q)]$. So by definition of φ^*_{n-1}, we have that $\varphi^*_{n-1}(\xi(q)) = (1, g_n(q)^{-1}f_n(q))$.

Thus, we have the equality

$$
\begin{aligned}
g_n(q)\phi^*_{n-1}(q) &= g_n(q)\pi_2 \circ \rho^*_{n-1}(\xi_n(q)) \\
&= g_n(q)\pi_2((1, g_n(q)^{-1}f_n(q)))
\end{aligned}
$$

$$= g_n(q)g_n(q)^{-1}f_n(q)$$
$$= f_n(q)$$

□

Proposition 5.5.3. Let $[\mathsf{H} : \partial]$ and $[\mathsf{G} : \delta]$ be reduced n-crossed complexes. The set function

$$\Omega : \pi_0 \mathbf{Rhom}(\mathsf{H}, \mathsf{G}) \to \pi_0 \mathsf{B}^n(\mathsf{H}, \mathsf{G})$$

is a surjection.

Proof. To show Ω is surjective, suppose $([\mathsf{E}, \eta], p, g, \alpha, \beta)$ is a n-butterfly. Folding the butterfly gives us the reduced n-crossed complex

$$\mathsf{B} : \mathsf{H}_n \times \mathsf{G}_n \xrightarrow{\alpha \times \beta} \mathsf{E}_{n-1} \longrightarrow \mathsf{E}_{n-2} \longrightarrow \cdots$$

where $\mathsf{E}_k = \mathsf{Q}_k$ for $k \leq n-2$. Also, we have the morphisms of reduced $(n-1)$-crossed complexes

$$\begin{tikzcd} & \mathsf{B} \arrow[dl, "p"'] \arrow[dr, "g"] & \\ \mathsf{H} & & \mathsf{G} \end{tikzcd}$$

Moreover, $p : \mathsf{B} \to \mathsf{H}$ is a trivial fibration by proposition 5.4.8.

Since Q is a cofibrant object, we have the lift l given in the commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \mathsf{B} \arrow[d, two heads, "p", "\simeq"'] \\ \mathsf{Q} \arrow[ur, dashed, "l"] \arrow[r, two heads, "\simeq", "p"'] & \mathsf{H} \end{tikzcd} \tag{5.14}$$

Composing the morphism l with $g : \mathsf{B} \to \mathsf{G}$, we have a morphism $\mathsf{Q} \to \mathsf{G}$ which we will denote by f. Thus, we have a derived morphism

$$\begin{tikzcd} & \mathsf{Q} \arrow[dl, two heads, "p"', "\simeq"] \arrow[dr, "f"] & \\ \mathsf{H} & & \mathsf{G} \end{tikzcd}$$

Moreover, we have the morphism $\nabla^f : \mathsf{Q} \to \mathsf{H} \times \mathsf{G}$ of reduced n-crossed complexes.

We claim that $\Omega(f)$ is in the same connected component as $([\mathsf{E}, \eta], q, g, \alpha, \beta)$. Taking the n-pushout, we have the factorization

$$\mathsf{Q} \longrightarrow \mathsf{Q}^{\nabla^f_n} \longrightarrow \mathsf{H} \times \mathsf{G}$$

We define a morphism from $\Theta : \mathsf{Q}^{\nabla^f_n}_{\leq n-1} \to \mathsf{E}_{\leq n-1}$. Since $\mathsf{Q}^{\nabla^f_n}_k = \mathsf{B}_k = \mathsf{Q}_k$ for $k \leq n-2$, we let $\Theta_k = 1_{\mathsf{Q}}$ for $k \leq n-2$.

By definition of $\nabla^f : \mathsf{Q} \to \mathsf{H} \times \mathsf{G}$, we have the commutative solid diagram

$$\begin{tikzcd}
 & \mathsf{H}_n \times \mathsf{G}_n \arrow[dr] \arrow[drr, bend left, "\alpha\times\beta"] & & \\
\mathsf{Q}_n \arrow[ur, "\nabla^f"] \arrow[dr, "\xi_n"'] & & \mathsf{Q}_{n-1} \times^{\nabla^f_n} \mathsf{H}_n \times \mathsf{G}_n \arrow[r, dashed, "\Theta_{n-1}"] & \mathsf{E}_{n-1} \\
 & \mathsf{Q}_{n-1} \arrow[ur] \arrow[urr, bend right, "l_{n-1}"'] & &
\end{tikzcd}$$

and the unique map Θ_{n-1} by the universal property of fiber coproducts.

By the two out of three property of model categories, $\mathsf{Q}^f_{n-1} \to \mathsf{B}$ is a trivial fibration. The five-lemma along with the commutative diagram of short exact sequences

$$\begin{tikzcd}
\mathbf{1} \arrow[r] \arrow[d] & \ker \xi^f_n \arrow[r] \arrow[d, "\cong"] & \mathsf{H}_n \times \mathsf{G}_n \arrow[r] \arrow[d, equal] & \mathsf{Q}^f_{n-1} \arrow[r] \arrow[d, dashed, "\Theta_{n-1}"] & \operatorname{coker} \xi^f_n \arrow[r] \arrow[d, "\cong"] & \mathbf{1} \arrow[d] \\
\mathbf{1} \arrow[r] & \ker \alpha \times \beta \arrow[r] & \mathsf{H}_n \times \mathsf{G}_n \arrow[r] & \mathsf{E}_{n-1} \arrow[r] & \operatorname{coker} \alpha \times \beta \arrow[r] & \mathbf{1}
\end{tikzcd}$$

implies that Θ_{n-1} is an isomorphism. Hence, the map $\Theta : \Omega(f) \to \mathsf{B}$ defines a morphism of n-butterflies where the homotopy is taken to be the trivial homotopy since the diagram

$$\begin{tikzcd}
\mathsf{Q}^{\nabla^f_n} \arrow[rr, "f"] \arrow[dr, "\Theta"'] & \arrow[d, phantom, "\Downarrow 1_f"] & \mathsf{G} \\
 & \mathsf{E}' \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes.

$\square$

The above three lemmas give the following main result.

Theorem 5.5.4. Let $[\mathsf{H} : \partial]$ and $[\mathsf{G} : \delta]$ be reduced n-crossed complexes. Then we have the bijection

$$\pi_0 \mathbf{Rhom}(\mathsf{H}, \mathsf{G}) \cong \pi_0 \mathsf{B}^n(\mathsf{H}, \mathsf{G})$$

APPENDIX A

MODEL CATEGORIES

After a theory in mathematics has been developed, the most natural question is whether the theory can be generalized in a manner that will produce more results and, possibly, be applied to other fields in some reasonable sense. This will be our task, focusing on homotopy theory, a particularly profitable theory in Algebraic Topology where one focuses on homotopies between continuous maps of spaces in an attempt to gain a procedure for classifying spaces. Of course, when generalizing this theory, the most appropriate category to work in is the category of topological spaces, **Top**. With the classical notion of homotopy, we are able to define homotopy equivalence and find homotopy invariants such as the fundamental group and higher homotopy groups using path homotopies. As these results have been very beneficial, we would like to be able to generalize the theory. Furthermore, it is natural to want a category in which we can view homotopy equivalent spaces as isomorphic. For example, S^1 as a subspace of $\mathbb{R}^2$ and $\mathbb{R}^2 - 0$ are homotopy equivalent with the inclusion map i (see Figure A.1) and a deformation retraction r (see Figure A.2) as homotopy inverses, but in **Top**, these spaces are obviously not isomorphic.

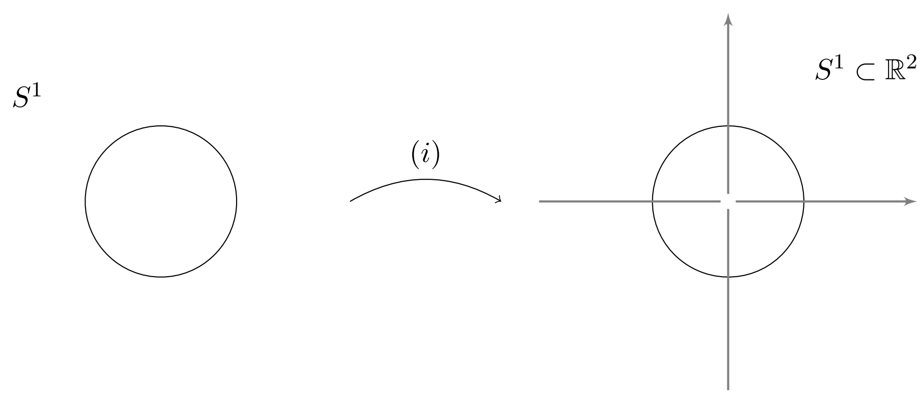

Figure A.1: Inclusion of S^1 into Plane

Thus, we would like to have a category where these two topological spaces are in fact isomorphic with r and i as isomorphisms i.e. $i \circ r = 1_{S^1}$ and $r \circ i = 1_{\mathbb{R}^2 - 0}$. There is already a procedure for producing such a category via categorical methods. Specifically, we can localize **Top** at the class of homotopy inverses which inverts all the homotopy inverses making them into isomorphisms. The

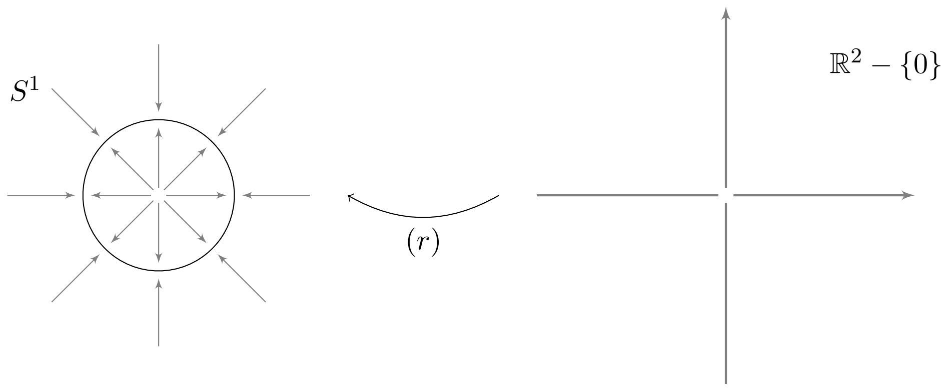

Figure A.2: Retract of Plane to S^1

result is a new category where the objects remain the same, but the morphisms are generated by the original morphisms of the category and the extra inverse morphisms. Unfortunately, since this procedure relies heavily on a universal property, there are no substantial tools for computing within the resulting category and, moreover, it is difficult to characterize. For example, it is not apparent that the resulting localization is even a locally small category which is necessary for most intents and purposes. Also, there are times when the notion of homotopy equivalence is to strong and we would prefer to focus on weaker notions such as weak homotopy equivalence. Daniel Quillen's model categories are categories with the necessary structure to define a homotopy theory with respect to a chosen class of *weak equivalences* and, furthermore, a well-defined, homotopy category. In fact, as one would hope, this homotopy category is equivalent to the localization of our model category with respect to the chosen class of morphisms, but, in contrast, is well equipped for calculations.

A particullarly enlightening example, being of an algebraic nature, is the category of chain complexes. Recall that homotopies of continuous maps in **Top** induce homotopies of chain maps in the category of chain complexes $\mathbf{Ch}(R)$. In fact, we have a notion of homotopy and homotopy equivalence in $\mathbf{Ch}(R)$ defined completely independent of the topological notion. Possibly of more importance than the homotopy equivalence of chain complexes is the weaker notion of quasi-isomorphism. With a suitable model structure, we can form the homotopy category of $\mathbf{Ch}(R)$ with respect to either the class of homotopy equivalences or the class of quasi-isomorphims. The homotopy category with respect to the latter class is equivalent to the derived category $\mathbf{D}(R)$, the localization of $\mathbf{Ch}(R)$ with respect to the quasi-isomorphisms. Again, we stress that since the derived category of chain complexes is constructed by a localization, it is not obvious that $\mathbf{D}(R)$ is locally small; however, the Quillen approach will guarantee such a claim.

We will give an overview of model categories and the construction of the homotopy category paying particular attention to examples. Then the focus will shift to defining a proper formulation of morphisms between model categories. For a more detailed and rigourous approach see [19] and [18].

A.1 General Definition

A.1.1 Prerequisites

Using machinery from category theory, we can reinvent some topological terminology into categorical terms. We will only define the necessary gadgets for defining a model structure, all of which can be found in either [18] or [19]. For the classical interpretations see [17].

Definition A.1.1. Let $f : A \to A'$ and $g : B \to B'$ in a category $\mathcal{C}$. Then f is a **retract** of g if there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "1_A"] & B \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ A' \arrow[r] \arrow[rr, bend right, "1_{A'}"'] & B' \arrow[r] & A' \end{tikzcd}$$

Definition A.1.2. Let the solid diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "l"] & D \end{tikzcd}$$

be commutative in a category $\mathcal{C}$. The morphism $i : A \to B$ has the **left lifting property** with respect to p and the morphism $p : C \to D$ has the **right lifting property** with respect to i if there is a morphism $l \in \mathcal{C}(B, C)$ which factors the diagram into commutative triangles.

For simplicity, the liftings above are commonly denoted by LLP or RLP, respectively.

Definition A.1.3. A **functorial factorization** of a category $\mathcal{C}$ is a pair of functors (i, p) : $\mathbf{Arr}(\mathcal{C}) \to \mathbf{Arr}(\mathcal{C})$ such that $f = p(f) \circ i(f)$ for all $f \in \mathbf{Arr}(\mathcal{C})$.

Definition A.1.4. In a pullback diagram

$$\begin{tikzcd} P \arrow[r, "u'"] \arrow[d, "v'"'] & B \arrow[d, "v"] \\ A \arrow[r, "u"'] & C \end{tikzcd} \tag{A.1}$$

the map u' is called the **base change** of u along v. Similarly, the map v' is called the base change of v along u.

Definition A.1.5. In a pushout diagram

$$\begin{tikzcd} C \arrow[r, "t"] \arrow[d, "s"'] & B \arrow[d, "s'"] \\ A \arrow[r, "t'"] & P \end{tikzcd} \tag{A.2}$$

the map s' is called the **cobase change** of s along t. Similarly, the map t' is called the cobase change of t along s.

A.1.2 General Definition

We now introduce the main structure of the section which was originally formulated by Daniel Quillen in [27].

Definition A.1.6. A **model category** is a category $\mathcal{M}$ with three closed subclasses of morphisms, called *weak equivalences*, *fibrations*, and *cofibrations*, which each contain the identity morphisms and satisfy the conditions **MC1-MC5**.

MC1 $\mathcal{M}$ is complete and cocomplete.

MC2 If $f, g, g \circ f \in \mathbf{Arr}(\mathcal{M})$ and two of the three maps are weak equivalences, then so is the third.

MC3 If $f, g \in \mathbf{Arr}(\mathcal{M})$ and f is a retract of g and g is a weak equivalence, fibration, or cofibration, then so is f, respectively.

MC4 Let the solid diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "l"] & D \end{tikzcd}$$

be commutative in $\mathcal{M}$. If i is a cofibration (trivial cofibration) and p is a trivial fibration (fibration), then there exists a lift l.

MC5 If $f \in \mathbf{Arr}(\mathcal{M})$, then there exists a functorial factorization (i, p) such that $i(f)$ is a cofibration, $p(f)$ is an trivial fibration, and a funtorical factorization (i', p') such that $i'(f)$ is an trivial cofibration, and $p'(f)$ is a fibration.

In diagrams, we denote *weak equivalences*, *fibrations*, and *cofibrations* by the arrows $\xrightarrow{\simeq}$, $\twoheadrightarrow$, and $\hookrightarrow$, respectively. A *trivial fibration* (*trivial cofribation*) is a morphism which is a fibration (cofibration) and a weak equivalence.

Remark A.1.7. A model category was originally called a *closed* model category to emphasize it has enough structure to guarantee that any two classes of morphisms determines the third, but conveniently "closed" has been dropped. Also, some definitions have the less stringent structure in which MC1 only requires finite limits and colimits, and the factorizations in MC5 do not have to be functorial. In most cases, including ours, this has no effect.

Theorem A.1.8. [13] Let $\mathcal{C}$ be a category, $\emptyset$ be the empty category, and $F : \emptyset \to \mathcal{C}$ be the unique functor. Then $\mathrm{colim} F$, if it exists, is the initial object of $\mathcal{C}$ and $\lim F$, if it exists, is the final object of $\mathcal{C}$.

Proof. Follows directly from the universality of colimit and limit, respectively. □

Corollary A.1.9. Let $\mathcal{M}$ be a model category. Then $\mathcal{M}$ has an initial object and final object.

We will denote the initial and final object of a model category by $\emptyset$ and $*$, respectively.

Example A.1.10. In **Top**, the initial object is the empty set and the terminal object is the one point space.

Example A.1.11. In $\mathbf{Ch}(R)$, the initial object and the terminal object are both the zero chain complex, $0_.$, which degree-wise is the zero R-module.

In cases when the initial object and terminal object agree, as in the above example, we call the unique object the *zero object*.

Definition A.1.12. Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}_0$. If the unique morphism $\emptyset \to X$ is a cofibration, then X is called a **cofibrant object**. If the unique morphism $X \to *$ is a fibration, then X is called a **fibrant object**.

The next theorem is quite useful when defining model categories because it implies that we only have to define weak equivalence, and either fibrations or cofibrations. In fact, there is a further result that any two of the three classes of morphisms completely characterize the third.

Theorem A.1.13. [18] Let $\mathcal{M}$ be a model category.

1. A map $i : A \to B$ is a cofibration (trivial cofibration) in $\mathcal{M}$ if and only if the map has the LLP with respect to trivial fibrations (fibrations).

2. A map $p : A \to B$ is a fibration (trivial fibration) in $\mathcal{M}$ if and only if the map has the RLP with respect to trivial cofibrations (cofibrations).

Proof. For 2, axiom MC4 states that having the RLP is a necessary condition. Thus, we need only prove that having the RLP with respect to trivial cofibrations (cofibrations) is a sufficient condition. Suppose we have the map $f : X \to Y$ having the RLP with respect to trivial cofibrations (cofibrations). Then by axiom MC5, f factors as $f = p \circ i$ where $i : X \to X'$ is a trivial cofibration (cofibration) and $p : X' \to Y$ is a fibration (trivial fibration). So the diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "i"'] & X \arrow[d] \\ X' \arrow[r, "p"] \arrow[ur, dashed, "l"] & Y \end{tikzcd}$$

commutes. Thus, by axiom MC4, there is a lift $l : X' \to X$. Now, since the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[r, "l"] \arrow[d, "p"] & X \arrow[d, "f"] \\ Y \arrow[r, "1"] & Y \arrow[r, "1"] & Y \end{tikzcd}$$

commutes, f is a retract of p. Hence, f is a fibration (trivial fibration). The argument for 1 follows by duality. $\square$

Theorem A.1.14. [13] Let $\mathcal{M}$ be a model category. Then the (trivial) fibrations in $\mathcal{M}$ are stable under base change and the (trivial) cofibrations are stable under cobase change.

A.1.3 Induced Model Categories

As one might have noticed, proving that a category is a model category is a difficult task so if we can find any shortcuts in our effort we should definitely exploit them. Some categories are constructed from others such as the dual category and comma categories. As we will see, there are model structures on these new categories which are induced from the category used to construct them.

Dual Model Category.

Definition A.1.15. Let $\mathcal{C}$ be a category. The **dual category** of $\mathcal{C}$, denoted by $\mathcal{C}^{op}$, has the same objects as $\mathcal{C}$ with the sets of morphisms defined by

$$\mathcal{C}^{op}(X, Y) = \mathcal{C}(Y, X) \quad \text{for} \quad X, Y \in \mathcal{C}^{op}.$$

When discussing categories, the dual category usually is a handy device, especially where contravariant functors appear. Thus, when considering model categories one would like to have an accessible model structure for the dual category.

Theorem A.1.16. Let $\mathcal{M}$ be a model category. Then the model structure on $\mathcal{M}$ induces a model structure on $\mathcal{M}^{op}$ by defining $f^{op} : Y \to X$ in $\mathcal{M}^{op}$ to be a *weak equivalence* if $f : X \to Y$ is a weak equivalence in $\mathcal{M}$, a *fibration* if $f : X \to Y$ is a cofibration in $\mathcal{M}$, and a *cofibration* if $f : X \to Y$ is a fibration in $\mathcal{M}$.

Remark A.1.17. Amending any property of $\mathcal{M}$ that depends on its model structure by simply flipping arrows, and interchanging fibrations and cofibrations will also hold for $\mathcal{M}^{op}$.

Comma Model Categories. Comma categories show up repeatedly and can be very useful; for example, the category of pointed topological spaces is a comma category constructed from **Top**.

Definition A.1.18. Let $\mathcal{C}$ be a category and $A \in \mathcal{C}_0$ be fixed. The **under comma category** is the category $\mathcal{C}^A$ where the objects are morphisms $A \to X$ for $X \in \mathcal{C}_0$ and the morphisms are commutative diagrams

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

Example A.1.19. Let $\mathcal{C} = \textbf{Top}$ and A a point space. Then the under comma category $\mathcal{C}^A$ is the category of pointed topological spaces.

In the case of comma categories, if a model structure is known for the base category, a model structure can be adapted to the induced category.

Theorem A.1.20. Let $\mathcal{M}$ be a model category. Then $\mathcal{M}$ induces a model structure on $\mathcal{M}^A$ by defining a morphism

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

to be a *weak equivalence* if $f : X \to Y$ is a weak equivalence in $\mathcal{M}$, a *fibration* if $f : X \to Y$ is a fibration in $\mathcal{M}$, and a *cofibration* if $f : X \to Y$ is a cofibration in $\mathcal{M}$.

There is a dual notion of the under category called the *over category* where arrows are reversed. We then have the dual result below.

Theorem A.1.21. The *over comma category* can be constructed in a similar manner and is denoted by $\mathcal{C}_A$. Moreover, a model category $\mathcal{M}$ will induce a model structure on $\mathcal{M}_A$.

A.1.4 Examples

Now, we give model structures on familiar categories beginning with the category of topological spaces. In fact, we will introduce two model structures on **Top** and remark A.1.28 will distinguish the two. As our main interests lie in algebra, we will also introduce two model structures on $\mathbf{Ch}_{\geq 0}(R)$ and $\mathbf{Ch}^{\geq 0}(R)$.

Hurewicz-Strøm Model on Top. The first model structure that we introduce on **Top** will result in producing the homotopy category of topological spaces where homotopy equivalences have been formally inverted. Before we define the structure, we will recall a couple of devices from algebraic topology.

Definition A.1.22. A map $p \in \mathbf{Top}(C, D)$ has the **homotopy lifting property** if for any topological space A and any commutative diagram

$$\begin{tikzcd} A \times 0 \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ A \times [0,1] \arrow[r] \arrow[ur, dashed, "l"] & D \end{tikzcd}$$

there exists a lift l. A morphism with the homotopy lifting property is a **Hurewicz fibration**.

Definition A.1.23. Let $A, B \in \mathbf{Top}_0$ and $A \subset B$. A map $i \in \mathbf{Top}(A, B)$ has the **homotopy extension property** if for every $Y \in \mathbf{Top}_0$ and commutative diagram

$$\begin{tikzcd} (B \times 0) \cup (A \times [0,1]) \arrow[r] \arrow[d] & Y \arrow[d] \\ B \times [0,1] \arrow[r] \arrow[ur, dashed, "l"] & * \end{tikzcd}$$

there exists a lift l. A map $i \in \mathbf{Top}(A, B)$ is a **closed Hurewicz cofibration** if A is a closed subspace of B and i has the homotopy extension property.

The Hurewicz-Strøm model structure on **Top** is as follows.

Theorem A.1.24. [19] A model structure exists on **Top** where $f \in \mathbf{Top}(X, Y)$ is a *weak equivalence* if f is a homotopy equivalence, a *fibration* if f is a Hurewicz fibration, and a *cofibration* if f is a closed Hurewicz cofibration.

Quillen Model on Top. Now, we define the more widely used model structure on **Top** where the weak equivalences are "weakened" and the fibrations are Hurcewiz fibrations, but restricted to CW-complexes. Thus, the focus is on CW-complexes. In some sense, this model structure was the prototype for defining model categories.

Definition A.1.25. A map $f \in \mathbf{Top}(X, Y)$ is a **weak homotopy equivalence** if for each basepoint $x \in X$ the map

$$f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is a bijection of pointed sets for $n = 0$ and an isomorphism of groups for $n \geq 1$.

Definition A.1.26. A map $p \in \mathbf{Top}(C, D)$ is a **Serre fibration** if for each CW-complex A and commutative diagram

$$\begin{tikzcd} A \times 0 \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ A \times [0,1] \arrow[r] \arrow[ur, dashed, "l"] & D \end{tikzcd}$$

there exists a lift l.

The Quillen model structure on **Top** is given by the following.

Theorem A.1.27. [19] A model structure exists on **Top** where $f \in \mathbf{Top}(X, Y)$ is a *weak equivalence* if f is a weak homotopy equivalence, a *fibration* if f is a Serre fibration, and a *cofibration* if f is a retract of a map $X \to Y'$ in which Y' obtained from X by attaching cells.

Remark A.1.28. To see that these two model structures are indeed different, notice that the morphism from the Warsaw Circle, the subspace of $\mathbb{R}^2$ obtained by connecting the interval [-1,1] on the y-axis and the curve $sin(1/x)$ on $0 < x \leq 1$ via an arc from the point $(0, -1)$ to $(1, sin(1))$ (see Figure A.3), to a point is a weak homotopy equivalence, but not a homotopy equivalence. Thus, the morphism is a weak equivalence in the Quillen model structure, but not in the Hurewicz-Strøm model structure.

The fibrant objects and the cofibrant objects are the cornerstones for constructing the homotopy category so we will discuss these objects in **Top** with the Quillen model structure.

Example A.1.29. The *fibrant objects* are the spaces X such that the unique map $X \to *$ is a fibration. So they are precisely the spaces X such that for every space A and every commutative diagram

$$\begin{tikzcd} A \times 0 \arrow[r, "k"] \arrow[d] & X \arrow[d, "p"] \\ A \times [0,1] \arrow[r] \arrow[ur, dashed, "l"] & * \end{tikzcd}$$

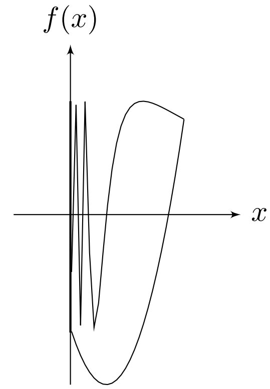

Figure A.3: Warsaw Circle

in **Top** there exists a lift l. Since the map $k \circ \pi_1 : A \times [0,1] \to X$ where π_1 is the projection onto A is a lift, every object in **Top** is fibrant. The *cofibrant objects* are the spaces X such that the unique map $\emptyset \to X$ is a cofibration. In other words, the map $\emptyset \to X$ is a retract of a map $\emptyset \to \emptyset'$ where $\emptyset'$ is the object obtained from $\emptyset$ by attaching cells. Since the intitial object in **Top** is the empty set, $\emptyset'$ is just a CW-complex. Thus, X is a cofibrant object precisely when X is a retract of some CW-complex. Moreover, since every CW-complex is a retract of itself, all CW-complexes are cofibrant objects.

Projective Model on $\mathbf{Ch}_{\geq 0}(R)$. In order to have an idea of the generality of a model category, we step away from the topological origins for a moment and we define two model structures on a purely algebraic category, specifically, the category of nonnegatively graded chain complexes over a ring R, $\mathbf{Ch}_{\geq 0}(R)$. First, we define the *projective model structure*.

Theorem A.1.30. [13] A model structure exists on $\mathbf{Ch}_{\geq 0}(R)$ where a morphism $f : X_{.} \to Y_{.}$ is a *weak equivalence* if f is a quasi-isomorphism, a *fibration* if f is an epimorphism in positive degrees, and a *cofibration* if f is a monomorphism with projective cokernel for all degrees.

Again, due to the importance of the fibrant and cofibrant objects, we will examine these objects in $\mathbf{Ch}_{\geq 0}(R)$.

Example A.1.31. The *fibrant objects* are the chain complexes $C_{.}$ such that the map $C_{.} \to 0_{.}$ is a fibration. Since the fibrations are just epimorphisms degree wise and every map from an R-module to the zero module is an epimorphism, every chain complex is fibrant. As for the *cofibrant objects*, these are simply the chain complexes $C_{.}$ such that the map $0_{.} \to C_{.}$ is a cofibration. Since the cofibrations are the monomorphisms with projective cokernels and the cokernel of the injective map

$0_\cdot \to C_\cdot$ is $C_\cdot$, the cofibrant objects are the chain complexes with projective R-modules in every degree.

We know introduce chain complexes which are analogs of the n-disk and n-sphere.

Definition A.1.32. The **n-disk chain complex** of a R-module A is defined by

$$D^n(A)_k = \begin{cases} 0 & k \neq n, n-1 \\ A & k = n, n-1 \end{cases}$$

for $n \geq 1$ where the boundary map is the identity in degree n and the zero map every where else.

Definition A.1.33. The **n-sphere chain complex** of a R-module A is defined by

$$S^n(A)_k = \begin{cases} 0 & k \neq n \\ A & k = n \end{cases}$$

for $n \geq 0$.

We compare $\mathbf{Ch}_{\geq 0}(R)$ with the more general category of unbounded chain complexes. We can generalize the projective model structure on $\mathbf{Ch}_{\geq 0}(R)$ to $\mathbf{Ch}(R)$; however, not all chain complexes with projective R-modules in every degree are cofibrant, as the next example illustrates.

Example A.1.34. [19] Let k be a field, $A = k[x]/(x^2)$ which is a projective k-module, and $A_\cdot$ be the chain complex with A in every degree and the differential is multiplication by x. Assume $A_\cdot$ is cofibrant in $\mathbf{Ch}(R)$. Since $A_\cdot$ is acyclic, $0_\cdot \to A_\cdot$ is a trivial cofibration. Since the natural map $A \to k$ is a surjection, the induced map $S^0(A) \to S^0(k)$ is a fibration. The surjection $A \to k$ also induces a map $A_\cdot \to S^0(k)$ in which the diagram

$$\begin{tikzcd} 0_\cdot \arrow[r] \arrow[d, hook, "\simeq"'] & S^0(A) \arrow[d, two heads] \\ A_\cdot \arrow[r] \arrow[ur, dashed, "l"] & S^0(k) \end{tikzcd}$$

commutes. With the generalized projective model structure on $\mathbf{Ch}(R)$, we see by MC4 there would exist a lift $l : A_\cdot \to S^0(A)$, but the lift in degree zero would have to be the identity on A which is not a chain map.

The following theorem will give a description of some of the chain complexes in $\mathbf{Ch}(R)$ that are cofibrant.

Theorem A.1.35. [19] Any bounded below chain complex of projective R-modules is cofibrant.

Injective Model on $\mathbf{Ch}^{\geq 0}(R)$. Now, we formulate the analog model structure for the non-negatively graded cochain complexes which is commonly known as the *injective model structure.*

Theorem A.1.36. A model structure exists for $\mathbf{Ch}^{\geq 0}(R)$ where a morphism $f : X^{\cdot} \to Y^{\cdot}$ is a *weak equivalence* if f is a quasi-isomorphism, a *fibration* if f is an epimorphism with injective kernel for all degrees, and a *cofibration* if f is a monomorphism in positive degrees.

A.2 Homotopies

For any model category, we will define the devices needed to construct a homotopy theory. In particular, we will generalize the notion of homotopy from topology using the machinery granted by the model structure. For an appropriate generalization of homotopy to the categorical setting, there are necessary objects in topology which guide our intuition on how to define homotopies of a model category.

We will first characterize what it means for a model category to have *path objects* and *cylinder objects.* At which point, the notion of right homotopy and its dual, left homotopy, become apparent. Homotopies will be where these notions overlap. All of which give even the most algebraic settings a nice geometrical interpretation.

Cylinder and Path Objects. The following definition is a bit of categorical language, but the name should be reminiscent of a common construction.

Definition A.2.1. Let $\mathcal{M}$ be a model category and $Y \in \mathcal{M}_1$. The **diagonal map** is the map $\Delta : Y \to Y \prod Y$ in the commutative diagram

$$\begin{tikzcd} & & Y \\ Y \arrow[urr, bend left, "1_Y"] \arrow[drr, bend right, "1_Y"'] \arrow[r, dotted, "\Delta"] & Y \prod Y \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & Y \end{tikzcd}$$

given by the universal property of products.

Remark A.2.2. In fact, since $\mathcal{M}$ is complete, this map is guaranteed to exist.

Definition A.2.3. Let $\mathcal{M}$ be a model category and $Y \in \mathcal{M}_0$. A **path object** of Y is any object P_Y such that there is a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\simeq", "i"'] \arrow[rr, bend left, "\Delta"] & P_Y \arrow[r, "p"'] & Y \coprod Y \end{tikzcd}$$

where i is a weak equivalence. A path object P_Y is a *good* path object if p is a fibration and a *very good* path object if p is a fibration and i is a cofibration.

The path object is by no means unique nor does it have to be the path space of some object as one might guess. However, the path space in **Top** is in fact a path object as we will now see.

Example A.2.4. Let **Top** be the model category with the Quillen model structure and $Y \in \mathbf{Top}_0$. Then the path space, Y^I, is a path object in **Top**. This can be seen by the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\simeq", "i"'] \arrow[rr, bend left, "\Delta"] & Y^I \arrow[r, "p"'] & Y \coprod Y \end{tikzcd}$$

where i is the map that sends each point to the constant path at that point and p is the map that sends each path to its end points. This obviously factors the diagonal map.

Example A.2.5. Let $\mathbf{Ch}_{\geq 0}(R)$ be the model category given and $M_. \in \mathbf{Ch}_{\geq 0}(R)$. Then by the proof of MC5 (1), the object $M_. \oplus P(M_. \prod M_.)$ is a very good path object in $\mathbf{Ch}_{\geq 0}(R)$.

Now, we define the dual of path object. We call on a bit more categorical language, but again the name hopefully is recognizable.

Definition A.2.6. Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}_0$. The **folding map** is the map $\nabla : X \coprod X \to X$ in the commutative diagram

$$\begin{tikzcd} X \arrow[dr, "i_1"'] \arrow[drr, bend left, "1_X"] & & \\ & X \coprod X \arrow[r, dotted, "\nabla"] & X \\ X \arrow[ur, "i_2"] \arrow[urr, bend right, "1_X"'] & & \end{tikzcd}$$

given by the universal property of coproducts.

Similarly to the diagonal map, since $\mathcal{M}$ is cocomplete, this map is guaranteed to exist.

Definition A.2.7. Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}_0$. A **cylinder object** of X is any object C_X such that there is a commutative diagram

$$\begin{tikzcd} X \coprod X \arrow[r, "i"'] \arrow[rr, bend left, "\nabla"] & C_X \arrow[r, "\simeq", "p"'] & X \end{tikzcd}$$

where p is a weak equivalence. A cylinder object C_X is a *good* cylinder object if i is a cofibration and a *very good* cylinder object if i is a cofibration and p is a fibration.

Again, the cylinder object is by no means unique nor does it have to be the cylinder of a category. However, the cylinder in **Top** and $\mathbf{Ch}(R)$ are in fact cylinder objects as we will now see.

Example A.2.8. Let **Top** be the model category with the Quillen model structure and $X \in \mathbf{Top}_0$. Then $X \coprod X = X \dot{\cup} X$ is the disjoint union and the folding map sends both parts of the disjoint union identically onto X. The folding map factors as

$$\begin{tikzcd} X \dot{\cup} X \arrow[r, "i"] \arrow[rr, bend left, "\nabla"] & X \times I \arrow[r, "p"] & X \end{tikzcd}$$

where $X \times I$ is the cylinder space of X, the map i sends each space X of the disjoint union to an end of $X \times I$ and p is the projection of $X \times I$ onto X which is a homotopy equivalence. Since a homotopy equivalence is a weak homotopy equivalence, the geometrical cylinder $X \times I$ defines a cylinder object as we hoped.

Example A.2.9. Consider the projective model structure on $\mathbf{Ch}_{\geq 0}(R)$. Let $M_. \in \mathbf{Ch}_{\geq 0}(R)$ and $1_{M_.} : C_. \to C_.$ be the identity chain map. Recall the homological mapping cylinder of $M_.$, $cyl(M_.)$, defined by

$$cyl(M_.)_n = M_n \oplus M_{n-1} \oplus M_n$$

with boundary

$$\partial_n((m_0, m_1, m_2)) = (\partial m_0 + m_1, -\partial m_1, \partial m_2 + m_1),$$

Let $i : M_. \oplus M_. \to cyl(M_.)$ be defined by $(m_0, m_1) \mapsto (m_0, 0, m_1)$ and $p : cyl(M_.) \to M_.$ be defined by $(m_0, m_1, m_2) \mapsto m_0 + m_2$. Then i and p are chain maps such that the diagram

$$\begin{tikzcd} X \oplus X \arrow[r, "i"'] \arrow[rr, bend left, "\nabla"] & cyl(M_.) \arrow[r, "p"'] & M_. \end{tikzcd}$$

commutes. Define the map $q : M_. \to cyl(M_.)$ by $q(m) = (0, 0, m)$. Then clearly $p \circ q = 1_{M_.}$ so $p \circ q$ is chain homotopic to $1_{M_.}$. Define a chain homotopy $\{s_k\}$ by $s((m_0, m_1, m_2)) = (0, m_0, 0)$. Then with a little calculation, we see that

$$1((m_0, m_1, m_2)) - qp((m_0, m_1, m_3)) = \partial s \ + \ s\partial$$

so $q \circ p$ is chain homotopic to $1_{cyl(M_.)}$. Since a chain homotopy equivalence is a quasi-isomorphism, p is a quasi-isomorphism. Hence, $cyl(M_.)$ is a cylinder object.

Right and Left Homotopy. With path objects and cylinder objects defined, we can now define right and left homotopies, respectively.

Definition A.2.10. Let $\mathcal{M}$ be a model category and $f, g \in \mathcal{M}(X, Y)$. The maps f, g are **right homotopic**, denoted by $f \simeq_r g$, if for some path object P_Y of Y, there exists a map $H : X \to P_Y$ such that the diagram

$$\begin{tikzcd} & P_Y \arrow[d, "p"] \\ X \arrow[ur, dotted, "H"] \arrow[r, "f \prod g"'] & Y \prod Y \end{tikzcd}$$

commutes. The map H is a *right homotopy* from f to g and if P_Y is a (*very*) *good* path object, then the map H is a (*very*) *good* right homotopy.

Lemma A.2.11. [13] Let $\mathcal{M}$ be a model category and $Y \in \mathcal{M}_0$. If Y is fibrant and P_Y is a good path object for Y, then the maps $\pi_1 \circ p, \pi_2 \circ p : P_Y \to Y$ where $\pi_1, \pi_2 : Y \prod Y \to Y$ are the natural projections are trivial fibrations.

Proof. Since P_Y is a good path object, we have the commutative diagram

$$\begin{tikzcd} & & & Y \\ Y \arrow[urrr, bend left, "1_Y"] \arrow[drrr, bend right, "1_Y"'] \arrow[r, "\simeq", "i"'] & P_Y \arrow[r, two heads, "p"'] & Y \prod Y \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & & Y \end{tikzcd}$$

Since the identity maps on Y are obviously weak equivalences, $\pi_1 \circ p, \pi_2 \circ p$ are weak equivalences by MC2. Since $Y \prod Y$ is defined by the diagram

$$\begin{tikzcd} Y \prod Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & Y \arrow[d] \\ Y \arrow[r] & * \end{tikzcd}$$

and Y is fibrant, π_1, π_2 are fibrations by A.1.14. Since p is a fibration, $\pi_1 \circ p, \pi_2 \circ p$ are fibrations by composition. Hence, $\pi_1 \circ p, \pi_2 \circ p$ are trivial fibrations. □

Lemma A.2.12. [13] Let $\mathcal{M}$ be a model category and $f \simeq_r g : X \to Y$. Then there exists a good right homotopy from f to g. If in addition X is cofibrant, then there exists a very good right homotopy from f to g.

Proof. Since $f \simeq_r g : X \to Y$ for some path object P_Y, there exists a right homotopy $H : X \to P_Y$ such that the solid diagram

$$\begin{tikzcd}
& P_Y \arrow[r, hook, dashed, "i'", "\simeq"'] \arrow[d, "p"] & P'_Y \arrow[dl, dashed, two heads, "p'"] \\
X \arrow[ur, "H"] \arrow[r, "f \prod g"'] & Y \prod Y &
\end{tikzcd} \tag{A.3}$$

commutes. By MC5, there exists the factorization (i', p') of p above. Thus, we have the diagram

$$\begin{tikzcd}
Y \arrow[r, "\simeq", "i"'] \arrow[rrr, bend left, "\Delta"] & P_Y \arrow[r, hook, "\simeq", "i'"'] & P'_Y \arrow[r, two heads, "p'"'] & Y \prod Y
\end{tikzcd} \tag{A.4}$$

where $i' \circ i$ is a weak equivalence by composition and p' is a fibration. Thus, P'_Y is a good cylinder object. Moreover, since the diagram A.3 commutes, $i' \circ H$ is the required right homotopy. Hence, there exists a good right homotopy from f to g.

As for the second part, suppose X is cofibrant. By the first part, there exists a good path homotopy H between f and g with a good path object P_Y. Thus, by MC5, we have the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "\simeq", "i"'] \arrow[rr, bend left, "\Delta"] \arrow[dr, "\simeq", "i'"'] & P_Y \arrow[r, two heads, "p"] & Y \prod Y \\
& P'_Y \arrow[u, two heads, "p'"'] &
\end{tikzcd}$$

Since $p \circ p'$ is a fibration by composition, P'_Y is a very good path object for Y. Furthermore, p' is a weak equivalence by MC2. Since X is cofibrant and the diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & P'_Y \arrow[d, two heads, "p'", "\simeq"'] \\
X \arrow[r, "H"] \arrow[ur, dashed, "H'"] & P_Y
\end{tikzcd}$$

commutes, there exists a lift $H' : X \to P'_Y$ by MC4. Thus, the diagram

$$\begin{tikzcd}
& P'_Y \arrow[d, "p'"] \\
& P_Y \arrow[d, "p"] \\
X \arrow[uur, "H'"] \arrow[ur, "H"] \arrow[r, "f \prod g"] & Y \prod Y
\end{tikzcd}$$

commutes and H' is the required right homotopy. Hence, there exists a very good right homotopy. $\square$

Theorem A.2.13. [18] Let $\mathcal{M}$ be a model category and $X, Y \in \mathcal{M}_0$ where Y is fibrant. Then the relation $\simeq_r$ is an equivalence relation on $\mathcal{M}(X, Y)$.

Now, we define left homotopy, the dual of right homotopy. As all of the results of right homotopy have dual results for left homotopy, we will merely mention the main results needed to move on.

Definition A.2.14. [14] Let $\mathcal{M}$ be a model category and $f, g \in \mathcal{M}(X, Y)$. The maps f, g are **left homotopic**, denoted by $f \simeq_l g$, if for some cylinder object C_X of X, there exists a map $H : C_X \to Y$ such that the diagram

$$\begin{tikzcd} X \coprod X \arrow[r, "f \coprod g"] \arrow[d, "i"'] & Y \\ C_X \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

commutes. The map H is said to be a *left homotopy* from f to g and if C_Y is a (*very*) *good* cylinder object, then the map H is a (*very*) *good* left homotopy.

Example A.2.15. Let $X, Y \in$ **Top** and $f, g : X \to Y$. Then f is homotopic in the classical sense if and only if there exists a map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ which is equivalent to saying the diagram

$$\begin{tikzcd} X \arrow[d, "i_0"'] \arrow[dr, "f"] & \\ X \times I \arrow[r, "F"] & Y \\ X \arrow[u, "i_1"] \arrow[ur, "g"'] & \end{tikzcd} \tag{A.5}$$

commutes. This is equivalent to the definition that f is left homotopic to g.

Example A.2.16. A very similar argument shows that the classical notion of homotopy in Homological Algebra is equivalent to left homotopy. The only difference is that you must show this degree-wise and make sure that everything commutes.

The following lemma follows by a similar procedure as for right homotopy.

Theorem A.2.17. [18] Let $\mathcal{M}$ be a model category and $X, Y \in \mathcal{M}_0$ where X is cofibrant. Then the relation $\simeq_l$ is an equivalence relation on $\mathcal{M}(X, Y)$.

Definition A.2.18. Let $\mathcal{M}$ be a model category. Two maps $f, g \in \mathcal{M}(X, Y)$ are homotopic, denoted by $f \simeq g$, if $f \simeq_l g$ and $f \simeq_r g$.

Theorem A.2.19. [18] Let $\mathcal{M}$ be a model category and $X, Y \in \mathcal{M}_0$ where X is cofibrant and Y is fibrant. Then the relation $\simeq$ is an equivalence relation on $\mathcal{M}(X, Y)$.

Proof. Since X is cofibrant and Y is fibrant, $\simeq_l$ and $\simeq_r$ are equivalence relations on $\mathcal{M}(X, Y)$. Homotopy is an equivalence relation follows immediately. $\square$

Theorem A.2.20. [13] Let $\mathcal{M}$ be a model category and $f, g \in \mathcal{M}(X, Y)$.

1. If X is cofibrant and $f \simeq_l g$, then $f \simeq_r g$.

2. If Y is fibrant and $f \simeq_r g$, then $f \simeq_l g$.

Proof. We prove the second claim and the first follows by duality. Since $f \simeq_r g$, there exists a good right homotopy $H : X \to P_Y$ where P_Y is a good path object for Y (see Lemma A.2.12) such that the diagram

$$\begin{tikzcd} Y \arrow[r, "\simeq", "i"'] \arrow[rr, bend left, "\Delta"] & P_Y \arrow[r, two heads, "p"'] & Y \coprod Y \end{tikzcd}$$

commutes. Since Y is fibrant and P_Y is a good path object, $\pi_1 \circ p, \pi_2 \circ p$ are trivial fibrations where $\pi_1, \pi_2 : Y \coprod Y \to Y$ are the natural projections (see Lemma A.2.11). By using MC2 and MC5, we can find a good cylinder object for X such that the diagram

$$\begin{tikzcd} X \coprod X \arrow[r, hook, "i'"'] \arrow[rr, bend left, "\nabla"] & C_X \arrow[r, "\simeq", "p'"'] & X \end{tikzcd}$$

commutes. Since the diagram

$$\begin{tikzcd} X \coprod X \arrow[r, "H \coprod (i \circ f)"] \arrow[d, hook, "i'"'] & P_Y \arrow[d, two heads, "\simeq"', "\pi_0 \circ p"] \\ C_X \arrow[r, "f \circ p'"'] \arrow[ur, dashed, "l"] & Y \end{tikzcd}$$

commutes, there exists a lift $l : C_X \to P_Y$. Moreover, $\pi_1 \circ p \circ l$ is the required left homotopy by the uniqueness of the universal property of products. $\square$

Corollary A.2.21. Let $\mathcal{M}$ be a model category and $X, Y \in \mathcal{M}_0$ where X is cofibrant and Y fibrant. Then there are bijection of sets

$$\begin{tikzcd} \mathcal{M}(X, Y)/\simeq_r \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & \mathcal{M}(X, Y)/\simeq_l \arrow[dl, "\cong"] \\ & \mathcal{M}(X, Y)/\simeq & \end{tikzcd}$$

Definition A.2.22. Let $\mathcal{M}$ be a model category. The **fibrant-cofibrant category** of $\mathcal{M}$ is category $\mathcal{M}_{cf}$ where the objects are the objects of $\mathcal{M}$ which are both fibrant and cofibrant and the morphism sets are defined by

$$\mathcal{M}_{cf}(X,Y) = \mathcal{M}(X,Y)$$

for objects X and Y of $\mathcal{M}_{cf}$.

The category $\mathcal{M}_{cf}$ is just the category $\mathcal{M}$ restricted to the objects which are both fibrant and cofibrant. So there is a *full embedding*[1] of the category $\mathcal{M}_{cf}$ into $\mathcal{M}$. Since every object of $\mathcal{M}_{cf}$ is both fibrant and cofibrant, homotopy is an equivalence relation on all the sets of morphisms. Moreover, as shown in [18], we have the following fact.

Theorem A.2.23. Let $\mathcal{M}$ be a model category. Then there is a category $\pi\mathcal{M}_{cf}$, referred to as the *classical homotopy category* of $\mathcal{M}$, where the objects are the same as $\mathcal{M}_{cf}$ and the morphisms are the homotopy classes of the morphisms in $\mathcal{M}_{cf}$. Composition of morphisms is induced by composition of morphisms in $\mathcal{M}_{cf}$.

A particularly interesting class of homotopies is the following class.

Definition A.2.24. A morphisms $f : X \to Y$ in a model category $\mathcal{M}$ is a **homotopy equivalence** if there exists a morphism $g : Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$.

With the general definition of homotopy equivalences in mind, recall the following theorem of algebraic topology.

The Classical Whitehead Theorem A.2.25. [17] Let X, Y CW-complexes and $f : X \to Y$ a continuous map which induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for $n \geq 0$. Then f is a homotopy equivalence.

Since every homotopy equivalence in **Top** is a weak homotopy equivalence, the above statement is actually an equivalence. Now, the following theorem gives a plausible reason for wanting to call the category above, the homotopy category.

The Whitehead Theorem for Model Categories A.2.26. [19] Let $\mathcal{M}$ be a model category and $f \in \mathcal{M}(X,Y)$ where X, Y are fibrant-cofibrant objects. Then f is a weak equivalence if and only if f is a homotopy equivalence.

[1] Categorically, a functor which is injective on the set of objects and fully faithful. Such functors can be seen as the categorical analog of set inclusions.

Unfortunately, in the process of defining $\pi\mathcal{M}_{cf}$, we have omitted quite a few objects, a cost which is unnecessary as we will see. In fact, we are not far from the answer.

Remark A.2.27. Although $\pi\mathcal{M}_{cf}$ may not be the right choice for the homotopy category of $\mathcal{M}$, we point out that all the weak equivalences were inverted and only the weak equivalences were inverted. Moreover, since a quotient of a set by an equivalence relation is still a set, indeed, $\pi\mathcal{M}_{cf}$ is a locally small category.

A.3 Homotopy Category

We will now construct the homotopy category and compare it to a purely categorical definition which was introduced in the introduction. The categorical definition is much simpler, but lacks the geometrical intuition that guides us in the prior construction and does not come with tools to study the category. Moreover, the categorical definition does not give any implication that the resulting homotopy category is a locally small category.

Constructive Homotopy Category. By now, it may have become apparent that objects that are fibrant and cofibrant have very nice properties. Unfortunately, not all the objects of a model category are of the sort; however, we will see that every object is weakly equivalent to a fibrant-cofibrant object. In particular, we will now analyze a procedure in which we can always replace an object with one which is fibrant and cofibrant.

Definition A.3.1. Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}_0$. A **fibrant replacement** of X is an object RX with a diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "\simeq"'] & & * \\ & RX \arrow[ur, two heads] & \end{tikzcd}$$

guaranteed by the functorial factorization $(\gamma,\ \delta)$ of MC5.
Similarly, a **cofibrant replacement** of X is an object QX in the diagram

$$\begin{tikzcd} \emptyset \arrow[rr] \arrow[dr, hook] & & X \\ & QX \arrow[ur, two heads, "\simeq"'] & \end{tikzcd}$$

guaranteed by the functorial factorization $(\alpha,\ \beta)$ of MC5.

Example A.3.2. [14] It is worth noting that in $Ch_{\geq 0}(R)$, if $S^0(M)$ is the chain complex that has a R-module M in degree zero and the trivial R-module in all other degrees, then a cofibrant replacement of $S^0(M)$ given by

$$\begin{tikzcd} 0 \arrow[rr, hook] \arrow[dr, hook, "i"'] & & S^0(M) \\ & Q(S^0(M)) \arrow[ur, two heads, "\simeq", "p"'] & \end{tikzcd}$$

is a projective resolution of M. To see this, note $Q(S^0(M))$ is of the form

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \ldots$$

where $P_i \in R\mathbf{Mod}$. Since cofibrations in $\mathbf{Ch}_{\geq 0}(R)$ are injective chain maps with projective cokernel and the cokernel of i is $Q(S^0(M))$, P_i is projective for all $i \geq 0$. Since $Q(S^0(M))$ is quasi-isomorphic to $S^0(M)$ and $S^0(M) = M[0]$, the complex $Q(S^0(M))$ is a projective resolution of $M[0]$.

Lemma A.3.3. [18] Let $\mathcal{M}$ be a model category and $f \in \mathcal{M}(X, Y)$. Then there exists a map $f^* : QRX \to QRY$ such that f is a weak equivalence if and only if f^* is a weak equivalence. Furthermore, the map f^* is unique up to homotopy.

Theorem A.3.4. Let $\mathcal{M}$ be a model category and $f \in \mathcal{M}(X, Y)$. Then there is a functor $QR : \mathcal{M} \to \pi\mathcal{M}_{cf}$ defined by $X \mapsto QRX$ and $f \mapsto [f^*]$ where $[f^*]$ is the equivalence class of f^* in $\pi\mathcal{M}_{cf}(QRX, QRY)$.

Definition A.3.5. Let $\mathcal{M}$ be a model category. The **homotopy category** of $\mathcal{M}$ is the category $Ho(\mathcal{M})$ where

$$Ho(\mathcal{M})_0 = \mathcal{M}_0$$

and

$$Ho(\mathcal{M})(X, Y) = \mathcal{M}(QRX, QRY)/\simeq .$$

Theorem A.3.6. [18] Let $\mathcal{M}$ be a model category, $X \in \mathcal{M}_0$ and $f \in \mathbf{Arr}(\mathcal{M})$. There is a functor $H_{\mathcal{M}} : \mathcal{M} \to Ho(\mathcal{M})$ defined by $X \mapsto X$ and $f \mapsto QR(f)$.

Furthermore, the functor $H_{\mathcal{M}}$ satisfies the following property which was our motivation.

Theorem A.3.7. [18] Let $\mathcal{M}$ be a model category and $f \in \mathbf{Arr}(\mathcal{M})$. Then $H_{\mathcal{M}}(f)$ is an isomorphism if and only if f is a weak equivalence.

Definition A.3.8. Let $\mathcal{M}$ be a model category and $X, Y \in \mathcal{M}$. The set of morphisms $Ho(\mathcal{M})(X, Y)$ is denoted by $[X, Y]_{\mathcal{M}}$.

The following theorem implies that the definition of the morphisms of the homotopy category can be relaxed.

Theorem A.3.9. [19] Let $\mathcal{M}$ be a model category and $X, Y \in \mathcal{M}$. Then there is an isomorphism $[X, Y]_{\mathcal{M}} = Ho(\mathcal{M})(X, Y) \cong \mathcal{M}(QX, RY)/\simeq$.

Lastly, we compare the classical homotopy category with the homotopy category defined above.

Theorem A.3.10. [19] Let $\mathcal{M}$ be a model category. The inclusion functor $\mathcal{M}_{cf} \to \mathcal{M}$ induces the equivalence of categories $\pi\mathcal{M}_{cf} \simeq Ho(\mathcal{M})$.

Non-constructive Homotopy Category. Now we formally define the categorical definition of the homotopy category that was introduced in the introduction. To do this, we will now define a localization of a category with respect to a specific class of morphisms. For a more detailed exposition, see [20].

Definition A.3.11. Let $\mathcal{C}$ be a category and $\mathcal{W} \subset \mathbf{Arr}(\mathcal{C})$. A **localization** of $\mathcal{C}$ with respect to $\mathcal{W}$ is the data of a big category $\mathcal{W}^{-1}\mathcal{C}$ and a functor $L : \mathcal{C} \to \mathcal{W}^{-1}\mathcal{C}$ satisfying:

1. $L(w)$ is an isomorphism for all $w \in \mathcal{W}$;
2. for any big category $\mathcal{D}$ and any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(w)$ is an isomorphism for all $w \in \mathcal{D}$, there exists a functor $U : \mathcal{W}^{-1}\mathcal{C} \to \mathcal{D}$ such that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "L"'] & \mathcal{D} \\ \mathcal{W}^{-1}\mathcal{C} \arrow[ur, "U"'] & \end{tikzcd}$$

   commutes up to isomorphism;
3. if U_1, U_2 are two objects of $\mathcal{D}^{\mathcal{W}^{-1}\mathcal{C}}$ then the natural map

$$\mathcal{D}^{\mathcal{W}^{-1}\mathcal{C}}(U_1, U_2) \to \mathcal{D}^{\mathcal{C}}(U_1 \circ L, U_2 \circ L)$$

   is bijective.

Example A.3.12. As an example in Homological Algebra, the localization of $\mathbf{Ch}_{\geq 0}(R)$ with respect to the class of quasi-isomorphisms is the derived category $\mathbf{D}(R)$.

Now, the categorical definition of the homotopy category in the introduction is simply the localization of a model category with respect to the weak equivalences. In other words, if $\mathcal{W}$ is the class of weak equivalences of a model category $\mathcal{M}$, then the categorical definition of the homotopy category is precisely $\mathcal{W}^{-1}\mathcal{M}$. We will make this more precise in the next section.

Equivalence. For model categories to fulfill their purpose, the homotopy category of a model category constructed homotopically must be isomorphic to the localization of the model category with respect to the class of weak equivalences.

Theorem A.3.13. [18] Let $\mathcal{M}$ be a model category. Then the natural functor $H_{\mathcal{M}} : \mathcal{M} \to Ho(\mathcal{M})$ is a localization of $\mathcal{M}$ with respect to the class of weak equivalences.

Corollary A.3.14. Let $\mathcal{M}$ be a model category and $\mathcal{W}$ be the class of weak equivalences. Then we have the equivalence

$$Ho(\mathcal{M}) \simeq \mathcal{W}^{-1}\mathcal{M}$$

of categories.

Proof. By the universality of localizations. $\square$

Example A.3.15. Since the derived category $\mathbf{D}(R)$ of chain complexes over R is a localization of $\mathbf{Ch}_{\geq 0}(R)$ with respect to the class of quasi-isomorphisms,

$$\mathbf{D}(R) \simeq Ho(\mathbf{Ch}_{\geq 0}(R))$$

where $Ho(\mathbf{Ch}_{\geq 0}(R))$ is the homotopy category with respect to the projective model structure.

A.4 Quillen Morphisms

Now that we have the desired categories in which to work, we would like to find the appropriate morphisms between them. These morphisms follow immediately after a little excursion into the construction of left and right derived functors. As their name may suggest, they root from the subject of Homological Algebra as we will see. After a discussion of their existence, we will see how they lead directly to the definition of our morphisms of model categories which we will call Quillen Functors.

A.4.1 Derived Functors

Definition A.4.1. Let $\mathcal{M}$ be a model category and $F : \mathcal{M} \to \mathcal{N}$ be a functor of categories. Then a **left derived functor** of F is a pair (LF, l) where LF is a functor and l is a natural transformation such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "H_{\mathcal{M}}"'] & \arrow[d, phantom, "\Uparrow l"] & \mathcal{N} \\ & Ho(\mathcal{M}) \arrow[ur, "LF"'] & \end{tikzcd}$$

commutes and if $(G,\, l')$ is any other such pair, there exists a natural transformation $t : G \to LF$ such that $l \circ (t \circ 1_{H_{\mathcal{M}}}) = l'$.

Similarly, a **right derived functor** of F is a pair $(RF,\, r)$ where RF is a funtor and r is a natural transformation such that the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "H_{\mathcal{M}}"'] & \arrow[d, phantom, "\Downarrow r"] & \mathcal{N} \\ & Ho(\mathcal{M}) \arrow[ur, "RF"'] & \end{tikzcd}$$

commutes and if $(G,\, r')$ is any other such pair, there exists a natural transformation $t : RF \to G$ such that $(t \circ 1_{H_{\mathcal{M}}}) \circ r = r'$.

It is worth noting that the universal properties imply that a left or right derived functor are unique up to unique isomorphism. Thus, from this point foward we will refer to the left and right derived functor.

Remark A.4.2. [18] As dealing with all the compositions of natural transformations may seem difficult, Hirschhorn gives a plausible figurative understanding of the universal properties of left and right derived functors. The left derived functor is a functor that is the closest to F on the left and the right derived functor is the closest functor to F on the right.

After defining left and right derived functors, we naturally lead to a discussion of total left and total right derived functors, as they are a particularly important case of left and right derived functors, respectively.

Definition A.4.3. Let $\mathcal{M}, \mathcal{N}$ be model categories and $F : \mathcal{M} \to \mathcal{N}$ a functor. The **total left derived functor** $(\mathbf{L}F, \mathbf{l})$ is the left derived functor of the composition $H_{\mathcal{N}} \circ F : \mathcal{M} \to Ho(\mathcal{N})$. Similarly, the **total right derived functor** $(\mathbf{R}F, \mathbf{r})$ is the right derived functor of the composition $H_{\mathcal{N}} \circ F$.

If we think of the derived category of chain complexes as the analog of the homotopy category, then one might begin to see the relevance of the terminology. In particular, the purpose of the total derived functors in Homological Algebra are to extend a functor to derived categories. In our case, we are extending a functor between model categories to their respective homotopy categories. Our mission is to find the sufficient conditions for the left and right derived functor to exist. In order to do this, we will first prove two lemmas.

Lemma A.4.4. [18] Let $\mathcal{M}$ be a model category and $F : \mathcal{M} \to \mathcal{N}$ be a functor.

1. Let X, Y be cofibrant objects in $\mathcal{M}$ and the map $f : X \to Y$ be a weak equivalence. Then f factors as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i"', "\simeq"] & & Y \\ & X' \arrow[ur, two heads, "p"', "\simeq"] & \end{tikzcd}$$

and there exists an trivial cofibration $q : Y \to X'$ such that $p \circ q = 1_Y$.

2. Let X, Y be fibrant objects in $\mathcal{M}$ and the map $f : X \to Y$ be a weak equivalence. Then f factors as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i"', "\simeq"] & & Y \\ & X' \arrow[ur, two heads, "p"', "\simeq"] & \end{tikzcd}$$

and there exists an trivial fibration $q : X' \to X$ such that $q \circ i = 1_X$.

Proof. We will prove the first part and the second follows by duality. Since X, Y are cofibrant, we have the commutative diagram

$$\begin{tikzcd} \emptyset \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "i_1"] \\ Y \arrow[r, "i_2"'] & X \coprod Y \end{tikzcd}$$

Thus, i_1, i_2 are trivial cofibrations by proposition A.1.14. By MC5, we have the factorization

$$\begin{tikzcd} X \coprod Y \arrow[rr, "f \coprod 1_Y"] \arrow[dr, hook, "k"'] & & Y \\ & Z \arrow[ur, two heads, "l"', "\simeq"] & \end{tikzcd}$$

Since cofibrations are closed under composition, $k \circ i_1$, $k \circ i_2$ are cofibrations. Since $f, l, 1_Y$ are weak equivalences, and $f = l \circ (k \circ i_1)$ and $1_Y = l \circ (k \circ i_2)$, we have that $k \circ i_1$ and $k \circ i_2$ are weak equivalences by MC2. Hence, letting $i = k \circ i_1, p = l, q = l \circ (k \circ i_2)$ and $X' = Z$, we have the desired claim. □

Corollary A.4.5. [18] Let $\mathcal{M}$ be a model category and $F : \mathcal{M} \to \mathcal{N}$ be a functor.

1. If F maps trivial cofibrations between cofibrant objects to isomorphisms, then F maps weak equivalences between cofibrant objects to isomorphisms.

2. If F maps trivial fibrations between fibrant objects to isomorphisms, then F maps weak equivalences between fibrant objects to isomorphisms.

Proof. Let X, Y be cofibrant objects and $f : X \to Y$ be a weak equivalence. Then f factors as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "\simeq", "i"'] & & Y \\ & X' \arrow[ur, two heads, "\simeq"', "p"] & \end{tikzcd}$$

and there exists a trivial cofibration $q : Y \to X'$ such that $p \circ q = 1_Y$ by the lemma above. Since X is cofibrant and $i : X \to X'$ is a cofibration, X' is cofibrant. By the hypothesis, $F(i)$ and $F(q)$ are isomorphisms. Since

$$\begin{aligned} F(f) &= F(p) \circ F(i) \\ &= F(p) \circ (F(q) \circ F(q^{-1})) \circ F(i) \\ &= (F(p) \circ F(q)) \circ F(q^{-1}) \circ F(i) \\ &= F(p \circ q) \circ F(q^{-1}) \circ F(i) \\ &= F(1_Y) \circ F(q^{-1}) \circ F(i) \\ &= F(q^{-1}) \circ F(i) \end{aligned}$$

Since a composition of isomorphisms is an isomorphism, $F(f)$ is an isomorphism. □

Theorem A.4.6. [18] Let $\mathcal{M}$ be a model category and $F : \mathcal{M} \to \mathcal{N}$ a functor.

1. If F maps trivial cofibrations between cofibrant objects to isomorphisms, then the left derived functor (LF, l) of F exists. Moreover, if X is cofibrant, l_X is an isomorphism.
2. If F maps trivial fibrations between fibrant objects to isomorphisms, then the right derived functor (RF, r) of F exists. Moreover, if X is fibrant, r_X is an isomorphism.

Proof. Let $X \in \mathcal{M}_0$ and $f \in \mathcal{M}(X, Y)$. Define $D : \mathcal{M} \to \mathcal{N}$ by $D(X) = F(QX)$ and $D(f) = F(Q(f))$ where Q is the cofibrant replacement functor. Since D is a composition of functors, D is a functor. If f is a weak equivalence in $\mathcal{M}$, then $Q(f)$ is a weak equivalence between cofibrant objects. Thus, there is a unique functor $Ho(\mathcal{M}) \to \mathcal{N}$ by the universal property of localizations. We will conveniently denote this functor by LF. Define a natural transformation $l : LF \circ H_{\mathcal{M}} \to F$ by $l(X) = F(i_X)$ where i_X is the natural weak equivalence between QX and X. Since $F(i_X) : F(QX) \to F(X)$ and $F(QX) = D(X) = LF \circ H_{\mathcal{M}}$, $l(X)$ is in fact a natural transformation from $LF \circ H_{\mathcal{M}}$ to F. Now, suppose (G, l') is a similar pair such that $G : Ho(\mathcal{M}) \to \mathcal{N}$ and $l' : G \circ H_{\mathcal{M}} \to F$. We need to find a natural transformation $t : G \circ H_{\mathcal{M}} \to LF \circ H_{\mathcal{M}}$ such that $l \circ (t \circ 1_{H_{\mathcal{M}}}) = l'$. Since l' is a natural transformation, $F(QX) = LF \circ H_{\mathcal{M}}$ and $F(i_X) = l(X)$, we have the commutative diagram

$$\begin{tikzcd}
(G \circ H_{\mathcal{M}})(QX) \arrow[r, "l'(QX)"] \arrow[d, "(G\circ H_{\mathcal{M}})(i_X)"'] & (LF \circ H_{\mathcal{M}})(X) \arrow[d, "l(X)"] \\
(G \circ H_{\mathcal{M}})(X) \arrow[r, "l'(X)"'] & F(X)
\end{tikzcd}$$

Since i_X is a weak equivalence, $(G \circ H_{\mathcal{M}})(i_X)$ is an isomorphism. Thus, let $t = l'(QX) \circ ((G \circ H_{\mathcal{M}})(i_X))^{-1}$. Since i_X is an trivial cofibration, by hypothesis $F(i_X)$ is an isomorphism. Thus t must be unique. $\square$

Corollary A.4.7. Let $\mathcal{M}, \mathcal{N}$ be model categories and $F : \mathcal{M} \to \mathcal{N}$ a functor.

1. If $H_{\mathcal{N}} \circ F$ maps trivial cofibrations between cofibrant objects to isomorphisms, then the total left derived functor $(\mathbf{L}F, l)$ of F exists.

2. If $H_{\mathcal{N}} \circ F$ maps trivial fibrations between fibrant objects to isomorphisms, then the total right derived functor $(\mathbf{R}F, r)$ of F exists.

To further the relevance between the left and right derived functors of model categories and the left and right derived functors in Homological Algebra we apply our new terminology to the tensor functor.

Example A.4.8. Let $\mathbf{Ch}(R)$ and $\mathbf{Ch}(\mathbb{Z})$ have the projective model structures and $M \in R\mathbf{Mod}$. Then we have the functor

$$\mathbf{Ch}(R) \xrightarrow{M\otimes_R -} \mathbf{Ch}(\mathbb{Z}) \xrightarrow{H} Ho(\mathbf{Ch}(\mathbb{Z}))$$

We first show that there exists the total left derived functor $\mathbf{L}(H \circ M \otimes_R -)$. By the corollary A.4.7, we need only show that $H \circ M \otimes_R -$ maps trivial cofibrations between cofibrant objects to isomorphisms in $Ho(\mathbf{Ch}(\mathbb{Z}))$. Suppose $i : A_. \to B_.$ is a trivial cofibration in $\mathbf{Ch}(R)$. Since i is injective degreewise, we have the short exact sequence

$$0_. \longrightarrow A_. \longrightarrow B_. \longrightarrow B/A \longrightarrow 0_.$$

Thus, we have a long exact sequence of homology groups. Since i is a quasi-isomorphism, B/A is acyclic. Also, since i is injective with projective cokernel for $n \geq 0$, $(B/A)_n$ is projective for all $n \geq 0$. It is known that $Z_n(B/A)$ is projective and

$$B/A \cong \oplus_{n\geq 1} D^n(Z_{n-1}(B/A)).$$

Thus,

$$B \cong A \oplus B/A \cong A \oplus (\oplus_{n\geq 1} D^n(Z_{n-1}(B/A))).$$

Since $M \otimes_R -$ commutes with direct sums,

$$(M \otimes_R -)(B) \cong (M \otimes_R -)(A) \oplus (\oplus_{n\geq 1}(M \otimes_R -)D^n(Z_{n-1}(B/A))).$$

Since $(M \otimes_R -)D^n(Z_{n-1}(B/A))$ is acyclic and homology commutes with direct sum, $H((M \otimes_R -)(B)) \cong H((M \otimes_R -)(A))$. Thus, $(M \otimes_R -)(i)$ is a weak equivalence. Since H maps weak equivalences to isomorphisms, $H \circ (M \otimes_R -)(i)$ is an isomorphism. Hence, $\mathbf{L}(H \circ M \otimes_R -)$ exists. Since the cofibrant replacement of $S^0(N)$ is a projective resolution $P_.$ of N and $S^0(N)$ is weakly equivalent to $P_.$, we have

$$\mathbf{L}(M \otimes_R -)(S^0(N)) \cong \mathbf{L}(M \otimes_R -)(P_.) \cong M \otimes_R P_.$$

Thus,

$$H_i(\mathbf{L}(M \otimes_R -)(S^0(N)) \cong H_i(M \otimes_R P_.) = Tor_i^R(M, N)$$

Theorem A.4.9. [13] Let $\mathcal{M}, \mathcal{N}$ be model categories and

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

be an adjoint pair[2]. If F preserves cofibrations and G preserves fibrations, then

$$Ho(\mathcal{M}) \underset{\mathbf{R}G}{\overset{\mathbf{L}F}{\rightleftarrows}} Ho(\mathcal{N})$$

are adjoints. Moreover, if $F(X) \to Y$ is a weak equivalence if and only if its adjoint morphism $X \to G(Y)$ is a weak equivalence for every cofibrant object $X \in \mathcal{M}_0$ and every fibrant object $Y \in \mathcal{N}_0$, , then $\mathbf{L}F$ and $\mathbf{R}G$ are inverse equivalences of categories.

A.4.2 Quillen Functors

Since the weak equivalences in a model category are precisely the isomorphisms in the homotopy category, it is quite easy to see that the best choice of morphisms between model categories would be precisely the ones that hold this structure. Moreover, these morphisms should certainly preserve constructions dependent on the model category structure such as cylinder objects, path objects, and homotopies. An "equivalence" of model categories should be a functor on model categories that induces an equivalence of homotopy categories. Theorem A.4.9 gives a complete description of such functors which we will now formally define.

[2] F is a left adjoint to G and G is a right adjoint to F.

Definition A.4.10. Let $\mathcal{M}, \mathcal{N}$ be model categories and

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

be an adjoint pair. Then F is a **left Quillen functor** if F preserves cofibrations and weak equivalences between cofibrant objects. Similarly, G is a **right Quillen functor** if G preserves fibrations and weak equivalences between fibrant objects. The pair (F, G) is called a **Quillen pair**.

Lemma A.4.11. [18] Let $\mathcal{M}, \mathcal{N}$ be model categories and

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

be a Quillen pair.

1. If X is a cofibrant object of $\mathcal{M}$ and CX is a cylinder object of X, then $F(CX)$ is a cylinder object for FX.

2. If Y is a fibrant object of $\mathcal{N}$ and PY is a path object of Y, then $G(PY)$ is a path object for GY.

Lemma A.4.12. [18] Let $\mathcal{M}, \mathcal{N}$ be model categories and

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

be a Quillen pair.

1. If $f, g : X \to Y$ are left homotopic maps in $\mathcal{M}$, then $F(f)$ and $F(g)$ are left homotopic in $\mathcal{N}$.

2. If $f, g : X \to Y$ are right homotopic maps in $\mathcal{N}$, then $G(f)$ and $G(g)$ are right homotopic in $\mathcal{M}$.

Theorem A.4.13. [18] Let $\mathcal{M}, \mathcal{N}$ be model categories and

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

be a Quillen pair. If X is a cofibrant object of $\mathcal{M}$ and Y is a fibrant object of $\mathcal{N}$, then the isomorphism

$$\mathcal{N}(FX, Y) \cong \mathcal{M}(X, GY)$$

induces an isomorphism

$$\mathcal{N}(FX, Y)/\simeq \;\; \cong \;\; \mathcal{M}(X, GY)/\simeq .$$

We now give the definition of the functors that give what we would like "equivalences" to be of model categories. Again, this definition follows from theorem A.4.9.

Definition A.4.14. Let $\mathcal{M}, \mathcal{N}$ be model categories and

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

be a Quillen pair. Then F is a **left Quillen equivalence** if for every cofibrant object $X \in \mathcal{M}_0$ and every fibrant object $Y \in \mathcal{N}_0$ $F(X) \to Y$ is a weak equivalence if and only if its adjoint morphism $X \to G(Y)$ is a weak equivalence. Similarly, G is a **right Quillen equivalence**. The pair (F, G) is called a **Quillen equivalence**.

Even though the next theorem is truly a restatement of theorem A.4.9, we state it anyhow to complete the idea.

Theorem A.4.15. [18] Let $\mathcal{M}, \mathcal{N}$ be model categories and

$$\mathcal{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{N}$$

be a Quillen pair. If (F, G) is a pair of Quillen equivalences, then the induced adjoint pair

$$Ho(\mathcal{M}) \underset{\mathbf{R}G}{\overset{\mathbf{L}F}{\rightleftarrows}} Ho(\mathcal{N})$$

form an equivalence of categories.

Example A.4.16. [14] Let $f \in \mathbf{CRing}(R, S)$ and res_f be the restriction of scalars functor. Then

$$\mathbf{Ch}(R) \underset{res_f}{\overset{S \otimes_R -}{\rightleftarrows}} \mathbf{Ch}(S)$$

is a Quillen pair with respect to the projective model structure. Moreover, if $R = S$, then this is a Quillen equivalence.

Example A.4.17. [27] Let $|-|$ be the geometric realization functor and $S(-)$ be the singular set functor. Then

$$\mathbf{sSet}_Q \underset{S(-)}{\overset{|-|}{\rightleftarrows}} \mathbf{Top}_Q$$

is a Quillen equivalence.

APPENDIX B

PROOFS

B.1 n-Pushout

Proposition B.1.1. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then the 1-pushout below f, $[\mathsf{H}^{f_1} : \partial^{f_1}]$, is a reduced crossed complex.

Proof. The 1-pushout below f gives precisely $[\mathsf{G} : \delta]$ which is a crossed complex by the hypothesis. □

Proposition B.1.2. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. For $n \geq 2$, the n-pushout below f, $[\mathsf{H}^{f_n} : \partial^{f_n}]$, is a chain complex which is abelian in degrees $k \geq 3$.

Proof. Let $n \geq 2$. Then the groups of the n-pushout, $\mathsf{H}_k^{f_n}$, are abelian for $k \geq 3$ since G_k, H_k and $\mathsf{H}_k \times^{\mathsf{H}_{k+1}} \mathsf{G}_{k+1}/\delta_{k+2}$ are abelian for $k \geq 3$.
Furthermore, for $k > n$ we have that

$$\begin{aligned} \partial_k^{f_n} \circ \partial_{k+1}^{f_n}(x) &= \partial_k^{f_n}\left(\partial_{k+1}^{f_n}(x)\right) \\ &= \delta_k(\delta_{k+1}(x)) \\ &= 1 \end{aligned}$$

and for $k < n-2$

$$\begin{aligned} \partial_k^{f_n} \circ \partial_{k+1}^{f_n}(x) &= \partial_k^{f_n}\left(\partial_{k+1}^{f_n}(x)\right) \\ &= \partial_k(\partial_{k+1}(x)) \\ &= 1 \end{aligned}$$

So in order to prove that $[\mathsf{H}^{f_n} : \partial^{f_n}]$ forms a chain complex, we must show that $\partial_k^{f_n} \circ \partial_{k+1}^{f_n} = 1$ for $k = n$, $k = n-1$, and $k = n-2$.

For $x \in \mathsf{H}_{n+1}^{f_n} = \mathsf{G}_{n+1}$, we have that

$$\begin{aligned} \partial_n^{f_n} \circ \partial_{n+1}^{f_n}(x) &= \partial_n^{f_n}\left(\partial_{n+1}^{f_n}(x)\right) \\ &= i_2 \circ q_{\delta_{n+1}}(\delta_{n+1}(x)) \\ &= i_2\left(q_{\delta_{n+1}}(\delta_{n+1}(x))\right) \end{aligned}$$

$$= i_2\left([1]\right)$$
$$= [1, [1]]$$

For $x \in \mathsf{H}_n^{f_n} = \mathsf{G}_n$, we have that

$$\begin{aligned}
\partial_{n-1}^{f_n} \circ \partial_n^{f_n}(x) &= \partial_{n-1}^{f_n}\left(\partial_n^{f_n}(x)\right) \\
&= \overline{\partial_{n-1} \circ \pi_1}\left(i_2 \circ q_{\delta_{n+1}}(x)\right) \\
&= \overline{\partial_{n-1} \circ \pi_1}\left([1, [x]]\right) \\
&= \partial_{n-1} \circ \pi_1\left((1, [x])\right) \\
&= \partial_{n-1}\left(\pi_1\left((1, [x])\right)\right) \\
&= \partial_{n-1}(1) \\
&= 1
\end{aligned}$$

For $[x, [y]] \in \mathsf{H}_{n-1}^{f_n} = \mathsf{H}_{n-1} \times^{\mathsf{H}_n} \mathsf{G}_n/\delta_{n+1}$, we have that

$$\begin{aligned}
\partial_{n-2}^{f_n} \circ \partial_{n-1}^{f_n}([x, [y]]) &= \partial_{n-2}^{f_n}\left(\partial_{n-1}^{f_n}([x, [y]])\right) \\
&= \partial_{n-2}\left(\overline{\partial_{n-1} \circ \pi_1}([x, [y]])\right) \\
&= \partial_{n-2}\left(\partial_{n-1} \circ \pi_1((x, [y]))\right) \\
&= \partial_{n-2}\left(\partial_{n-1}\left(\pi_1((x, [y]))\right)\right) \\
&= \partial_{n-2}\left(\partial_{n-1}(x)\right) \\
&= 1
\end{aligned}$$

Thus, for $n \geq 2$, $[\mathsf{H}^{f_n} : \partial^{f_n}]$ is a chain complex which is abelian in degrees $k \geq 3$. $\square$

Proposition B.1.3. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then the 2-pushout below f, $[\mathsf{H}^{f_2} : \partial^{f_2}]$, is a reduced crossed complex.

Proof. The 2-pushout below f, $[\mathsf{H}^{f_2} : \partial^{f_2}]$, is of the form

$$\cdots \longrightarrow \mathsf{G}_4 \longrightarrow \mathsf{G}_3 \longrightarrow \mathsf{G}_2 \longrightarrow \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$$

By B.1.2, $[\mathsf{H}^{f_2} : \partial^{f_2}]$, is a chain complex which is abelian in degrees $k \geq 3$.
The action of $\mathsf{H}_1^{f_2}$ on $\mathsf{H}_2^{f_2}$ is defined in proposition 5.2.8 where it is also shown that

$$\partial^{f_2} : \mathsf{G}_2 \to \mathsf{H}_1 \ltimes^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$$

with this action forms a crossed module.

Now, for $k \geq 3$, let $x \in \mathsf{H}_k^{f_2} = \mathsf{G}_k$. Then the action of $[a,[b]]$ on x is the action defined by $x^{[a,[b]]} = x^{f_1(a)}$.

We first make sure that this action is defined on the quotient $\mathsf{H}_1 \times^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$. Suppose $z \in \mathsf{H}_2$. Then we have

$$\begin{aligned} x^{\left(\partial_2(z)^{-1}, q_{\delta_3} \circ f_2(z)\right)} &= x^{f_1\left(\partial_2(z)^{-1}\right)} \\ &= x^{f_1\left(\partial_2\left(z^{-1}\right)\right)} \\ &= x^{\delta_2\left(f_2\left(z^{-1}\right)\right)} \\ &= x \end{aligned}$$

since G is a crossed complex and the image of δ_2 acts trivially on G_k for $k \geq 3$.

Furthermore, by definition of this action, the image of $\partial_2^{f_n} : \mathsf{G}_2 \to \mathsf{H}_1 \times^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$ acts trivially on $\mathsf{H}_k^{f_n}$ for $k \geq 3$. Indeed, the elements of the image are of the form $[1,[x]]$ where $x \in \mathsf{G}_2$. So for $k \geq 3$ and $z \in \mathsf{H}_k^{f_n} = \mathsf{G}_k$

$$z^{[1,[x]]} = z^{f_1(1)} = z^1 = z$$

Now, we check that, for $k \geq 3$, $\partial_k^{f_n}$ commutes with the action. Since $\partial_2^{f_2} : \mathsf{G}_2 \to \mathsf{H}_1 \times^{\mathsf{H}_2} \mathsf{G}_2/\delta_3$ is a crossed module, property CM1 says $\partial_2^{f_2}$ commutes with the action. For the case when $k = 3$, since the action of G_1 commutes with δ_3, we have that

$$\begin{aligned} \partial_3^{f_n}(x)^{[a,[b]]} &= \delta_3(x)^{[a,[b]]} \\ &= b^{-1}\delta_3(x)^{f_1(a)} b \\ &= b^{-1}\delta_3\left(x^{f_1(a)}\right) b \end{aligned}$$

By property CM2 of δ_2, we have

$$\begin{aligned} b^{-1}\delta_3\left(x^{f_1(a)}\right) b &= \delta_3\left(x^{f_1(a)}\right)^{\delta_2(b)} \\ &= \delta_3\left(\left(x^{f_1(a)}\right)^{\delta_2(b)}\right) \end{aligned}$$

Since the image of δ_2 acts trivially on G_3,

$$\begin{aligned} \delta_3\left(\left(x^{f_1(a)}\right)^{\delta_2(b)}\right) &= \delta_3\left(x^{f_1(a)}\right) \\ &= \delta_3\left(x^{[a,[b]]}\right) \\ &= \partial_3^{f_n}\left(x^{[a,[b]]}\right) \end{aligned}$$

For $k > 3$, we have

$$\begin{aligned}
\partial_k^{f_n}(x)^{[a,[b]]} &= \delta_k(x)^{[a,[b]]} \\
&= \delta_k(x)^{f_1(a)} \\
&= \delta_k\left(x^{f_1(a)}\right) \\
&= \delta_k\left(x^{[a,[b]]}\right) \\
&= \partial_k^{f_n}\left(x^{[a,[b]]}\right)
\end{aligned}$$

Thus, $\partial_k^{f_2}$ commutes with the action of $\mathsf{H}_1^{f_2}$.

Hence, $[\mathsf{H}^{f_2} : \partial^{f_2}]$ is a reduced crossed complex. □

The next proposition requires proposition 4.5.9. Both of which I believe are known, but can be easily proved.

Proposition B.1.4. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. For $n \geq 3$, the n-pushout below f, $[\mathsf{H}^{f_n} : \partial^{f_n}]$, is a reduced crossed complex.

Proof. For the case when $n \geq 3$, $[\mathsf{H}^{f_n} : \partial^{f_n}]$ has the form

$$\cdots \longrightarrow \mathsf{G}_4 \longrightarrow \mathsf{G}_3 \longrightarrow \mathsf{H}_2 \times^{\mathsf{H}_3} \mathsf{G}_3/\delta_4 \longrightarrow \mathsf{H}_1$$

For the case when $n \geq 4$, $[\mathsf{H}^{f_n} : \partial^{f_n}]$ has the form

$$\cdots \longrightarrow \mathsf{G}_{n+1} \longrightarrow \mathsf{G}_n \longrightarrow \mathsf{H}_{n-1} \times^{\mathsf{H}_n/\partial_{n+1}} \mathsf{G}_n/\delta_{n+1} \longrightarrow \mathsf{H}_{n-2} \longrightarrow \cdots \longrightarrow \mathsf{H}_2 \longrightarrow \mathsf{H}_1$$

By B.1.2, for $n \geq 3$, $[\mathsf{H}^{f_n} : \partial^{f_n}]$ is a chain complex which is abelian in degrees $k \geq 3$.

Now, we describe the action of $\mathsf{H}_1^{f_n}$ on $\mathsf{H}_k^{f_n}$ for $k \geq 1$. The action on itself is just conjugation.

Let $a \in \mathsf{H}_1^{f_n} = \mathsf{H}_1$.

For $k < n-1$, let $x \in \mathsf{H}_k^{f_n} = \mathsf{H}_k$. Then we define the action of a on x by the given action of H.

For $k = n-1$, let $[x,[y]] \in \mathsf{H}_k^{f_n} = \mathsf{H}_{n-1} \times^{\mathsf{H}_n} \mathsf{G}_n/\delta_{n+1}$. Then we define the action of a on $[x,[y]]$ by $[x,[y]]^a = \left[x^a, [y]^{f_1(a)}\right]$. The action of G_1 on $\mathsf{G}_n/\delta_{n+1}$ is given by $[y]^{f_1(a)} = \left[y^{f_1(a)}\right]$. Let $z \in \mathsf{H}_n$. Then

$$\begin{aligned}
[1,[1]]^a &= \left(\partial_n(z)^{-1}, q_{\delta_{n+1}} \circ f_n(z)\right)^a \\
&= \left(\partial_n(z)^{-1}, q_{\delta_{n+1}}\left(f_n(z)\right)\right)^a \\
&= \left(\left(\partial_n(z)^{-1}\right)^a, q_{\delta_{n+1}}\left(f_n(z)\right)^{f_1(a)}\right)
\end{aligned}$$

Since ∂_n commutes with the action of H_1, $q_{\delta_{n+1}}$ is 1_{G_1}-invariant, and f_n is f_1-invariant, we have that

$$\begin{aligned}
\left(\left(\partial_n(z)^{-1}\right)^a, q_{\delta_{n+1}}\left(f_n(z)\right)^{f_1(a)}\right) &= \left(\partial_n(z^a)^{-1}, q_{\delta_{n+1}}\left(f_n(z)\right)^{1_{G_1}(f_1(a))}\right) \\
&= \left(\partial_n(z^a)^{-1}, q_{\delta_{n+1}}\left(f_n(z)^{f_1(a)}\right)\right) \\
&= \left(\partial_n(z^a)^{-1}, q_{\delta_{n+1}}\left(f_n(z^a)\right)\right) \\
&= [1, [1]]
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left([x,[y]]^a\right)^b &= \left[x^a, [y]^{f_1(a)}\right]^b \\
&= \left[x^{ab}, [y]^{f_1(a)f_1(b)}\right] \\
&= \left[x^{ab}, [y]^{f_1(ab)}\right] \\
&= [x,[y]]^{ab}
\end{aligned}$$

and

$$\begin{aligned}
\left([x,[y]][u,[v]]\right)^a &= [xu, [y][v]]^a \\
&= \left[(xu)^a, ([y][v])^{f_1(a)}\right] \\
&= \left[x^a u^a, [y]^{f_1(a)}[v]^{f_1(a)}\right] \\
&= \left[x^a, [y]^{f_1(a)}\right]\left[u^a, [v]^{f_1(a)}\right] \\
&= [x,[y]]^a [u,[v]]^a
\end{aligned}$$

Thus, the action defined above is a well-defined action on the group $\mathsf{H}_{n-1} \times^{\mathsf{H}_n} \mathsf{G}_n/\delta_{n+1}$.

For $k > n-1$, let $x \in \mathsf{H}_k^{f_n} = \mathsf{G}_k$. Then we define the action of a on x by the usual action given by f, specifically, $x^a = x^{f_1(a)}$.

Now, we check that $\partial_2^{f_n} : \mathsf{H}_2^{f_n} \to \mathsf{H}_1^{f_n}$ is a crossed module for $n \geq 3$.

For $n = 3$, $\partial_2^{f_n}$ has the form

$$\overline{\partial_2 \circ \pi_1} : \mathsf{H}_2 \times^{\mathsf{H}_3} \mathsf{G}_3/\delta_4 \to \mathsf{H}_1$$

We must check that $\partial_2^{f_3}$ satisfies the properties CM1 and CM2 of crossed modules.

For CM1, we have

$$\begin{aligned}
\partial_2^{f_n}\left([x,[y]]^a\right) &= \overline{\partial_2 \circ \pi_1}\left([x,[y]]^a\right) \\
&= \overline{\partial_2 \circ \pi_1}\left(\left[x^a, [y]^{f_1(a)}\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&= \partial_2 \circ \pi_1 \left(\left(x^a, [y]^{f_1(a)} \right) \right) \\
&= \partial_2 \left(x^a \right)
\end{aligned}
$$

Since $\partial_2 : \mathsf{H}_2 \to \mathsf{H}_1$ is a crossed module,

$$
\begin{aligned}
\partial_2 \left(x^a \right) &= a^{-1} \partial_2 \left(x \right) a \\
&= a^{-1} \partial_2 \circ \pi_1 \left((x, [y]) \right) a \\
&= a^{-1} \overline{\partial_2 \circ \pi_1} \left([x, [y]] \right) a \\
&= a^{-1} \partial_2^{f_n} \left([x, [y]] \right) a
\end{aligned}
$$

For CM2, we have

$$
\begin{aligned}
[x, [y]]^{\partial_2^{f_n}([u,[v]])} &= [x, [y]]^{\overline{\partial_2 \circ \pi_1}([u,[v]])} \\
&= [x, [y]]^{\partial_2 \circ \pi_1((u,[v]))} \\
&= [x, [y]]^{\partial_2(\pi_1((u,[v])))} \\
&= [x, [y]]^{\partial_2(u)} \\
&= \left[x^{\partial_2(u)}, [y]^{f_1(\partial_2(u))} \right] \\
&= \left[x^{\partial_2(u)}, [y]^{\delta_2(f_2(u))} \right] \\
&= \left[x^{\partial_2(u)}, \left[y^{\delta_2(f_2(u))} \right] \right]
\end{aligned}
$$

Since ∂_2 lifts to conjugation and the image of δ_2 acts trivially on G_3/δ_4,

$$
\begin{aligned}
\left[x^{\partial_2(u)}, \left[y^{\delta_2(f_2(u))} \right] \right] &= \left[u^{-1} x u, [y] \right] \\
&= \left[u^{-1} x u, [v]^{-1} [v] [y] \right] \\
&= \left[u^{-1} x u, [v]^{-1} [y] [v] \right]
\end{aligned}
$$

Since G_3 is abelian,

$$
\begin{aligned}
\left[u^{-1} x u, [v]^{-1} [y] [v] \right] &= \left[u^{-1} x u, [v]^{-1} [y] [v] \right] \\
&= \left[u^{-1}, [v]^{-1} \right] [x, [y]] \, [u, [v]]
\end{aligned}
$$

For $n \geq 4$, $\partial_2^{f_n}$ has the form

$$
\partial_2 : \mathsf{H}_2 \to \mathsf{H}_1
$$

which is a crossed module since H is a crossed complex. Now, we show that for $n \geq 3$ the image of $\partial_2^{f_n}$ acts trivially on $\mathsf{H}_k^{f_n}$ for $k \geq 3$.

Suppose $n = 3$. Let $[a,[b]] \in \mathsf{H}_2^{f_3} = \mathsf{H}_2 \times^{\mathsf{H}_3} \mathsf{G}_3/\delta_4$ and for $k \geq 3$ let $x \in \mathsf{H}_k^{f_3} = \mathsf{G}_k$. Then by the definition of the action of H_1 on G_k for $k \geq 3$, we have that

$$\begin{aligned}
x^{\partial_2^{f_n}([a,[b]])} &= x^{f_1\left(\partial_2^{f_n}([a,[b]])\right)} \\
&= x^{f_1\left(\overline{\partial_2 \circ \pi_1}([a,[b]])\right)} \\
&= x^{f_1(\partial_2 \circ \pi_1((a,[b])))} \\
&= x^{f_1(\partial_2(\pi_1((a,[b]))))} \\
&= x^{f_1(\partial_2(a))} \\
&= x^{\delta_2(f_2(a))} \\
&= x
\end{aligned}$$

since the image of δ_2 acts trivially on G_3.

Suppose $n \geq 4$. Let $a \in \mathsf{H}_2^{f_n} = \mathsf{H}_2$. We have three cases:

Case 1: For $3 \leq k < n-1$, let $x \in \mathsf{H}_k^{f_n} = \mathsf{H}_k$. In this case the action of $\mathsf{H}_k^{f_n} = \mathsf{H}_1$ on $\mathsf{H}_k^{f_n} = \mathsf{H}_k$ is the inherited one from the crossed complex H. So this acts trivially.

Case 2: For $k = n-1$, let $[a,[b]] \in \mathsf{H}_k^{f_n} = \mathsf{H}_{n-1} \times^{\mathsf{H}_n} \mathsf{G}_n/\delta_{n+1}$. Then we have that

$$\begin{aligned}
[a,[b]]^{\partial_2^{f_n}(a)} &= [a,[b]]^{\partial_2(a)} \\
&= [a^{\partial_2(a)}, [b]^{f_1(\partial_2(a))}] \\
&= [a^{\partial_2(a)}, [b]^{\delta_2(f_2(a))}]
\end{aligned}$$

since the image of ∂_2 acts trivially on H_k for $k \geq 3$ and δ_2 acts trivially on G_k for $k \geq 3$.

Case 3: For $k > n-1$, let $x \in \mathsf{H}_k^{f_n} = \mathsf{G}_k$. Then by definition of the action of H_1 on G_k and since the image of δ_2 acts trivially on G_k for $k \geq 3$, we have that

$$\begin{aligned}
x^{\partial_2^{f_2}(a)} &= x^{\partial_2(a)} \\
&= x^{f_1(\partial_2(a))} \\
&= x^{\delta_2(f_2(a))} \\
&= x
\end{aligned}$$

Lastly, we check that the action commutes with the differential $\partial_k^{f_n}$. Let $a \in \mathsf{H}_1$

For $k < n-1$, we have

$$\partial_k^{f_n} = \partial_k : \mathsf{H}_k \to \mathsf{H}_{k-1}$$

and this homomorphism commutes with the action of H_1 since H is a crossed complex.

For $k = n-1$, since ∂_k commutes with the action of H_1, we have

$$\begin{aligned}
\partial_k^{f_n}\left([x,[y]]^a\right) &= \overline{\partial_k \circ \pi_1}\left([x,[y]]^a\right) \\
&= \overline{\partial_k \circ \pi_1}\left(\left[x^a,[y]^{f_1(a)}\right]\right) \\
&= \partial_k \circ \pi_1\left(\left(x^a,[y]^{f_1(a)}\right)\right) \\
&= \partial_k\left(\pi_1\left(\left(x^a,[y]^{f_1(a)}\right)\right)\right) \\
&= \partial_k\left(x^a\right) \\
&= \partial_k(x)^a \\
&= \partial_k\left(\pi_1\left((x,[y])\right)\right)^a \\
&= \partial_k \circ \pi_1\left((x,[y])\right)^a \\
&= \overline{\partial_k \circ \pi_1}\left([x,[y]]\right)^a \\
&= \partial_k^{f_n}\left([x,[y]]\right)^a
\end{aligned}$$

For $k = n$, since we know q_{δ_2} is 1_{G_1}-invariant from 4.5.9, we have

$$\begin{aligned}
\partial_k^{f_n}\left(x^a\right) &= \partial_k^{f_n}\left(x^{f_1(a)}\right) \\
&= \bar{i}_2 \circ q_{\delta_n+1}\left(x^{f_1(a)}\right) \\
&= \bar{i}_2\left(q_{\delta_n+1}\left(x^{f_1(a)}\right)\right) \\
&= \bar{i}_2\left(\left[x^{f_1(a)}\right]\right) \\
&= \left[1,\left[x^{f_1(a)}\right]\right] \\
&= \left[1,[x]^{f_1(a)}\right] \\
&= \left[1^a,[x]^{f_1(a)}\right] \\
&= [1,[x]]^a \\
&= \bar{i}_2\left([x]\right)^a \\
&= \bar{i}_2\left(q_{\delta_{n+1}}(x)\right)^a \\
&= \bar{i}_2 \circ q_{\delta_{n+1}}(x)^a \\
&= \partial_k^{f_n}(x)^a
\end{aligned}$$

For $k > n$, by the definition of the action of H_1 on G_k and since δ_k commutes with the action of G_1, we have

$$\partial_k^{f_n}(x^a) = \delta_k\left(x^{f_1(a)}\right)$$

$$= \delta_k\,(x)^{f_1(a)}$$

$$= \delta_k\,(x)^{a}$$

$$= \partial_k^{f_n}\,(x)^{a}$$

Hence, $\left[\mathsf{H}^{f_n} : \partial^{f_n}\right]$ is a reduced crossed complex for $n \geq 2$. $\square$

B.2 n-Factorization

Proposition B.2.1. Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism of reduced crossed complexes. Then $[\mathsf{H}^{f^n} : \partial^{f^n}]$ factors f.

Proof. In general, we define the factorization by

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\mathsf{H}_{n+1} \arrow[r, "f_{n+1}"] \arrow[d] & \mathsf{G}_{n+1} \arrow[r, equal] \arrow[d] & \mathsf{G}_{n+1} \arrow[d] \\
\mathsf{H}_n \arrow[r, "f_n"] \arrow[d, "\partial_n"'] & \mathsf{G}_n \arrow[r, equal] \arrow[d, "\bar{i}_2 \circ q_{\delta_{n+1}}"] & \mathsf{G}_n \arrow[d, "\delta_n"] \\
\mathsf{H}_{n-1} \arrow[r, "\bar{i}_1"] \arrow[d, "\partial_{n-1}"'] & \mathsf{H}_{n-1} \times^{\mathsf{H}_n} \mathsf{G}_n/\delta_{n+1} \arrow[r, "\rho"] \arrow[d, "\overline{\partial_{n-1} \circ \pi_1}"] & \mathsf{G}_{n-1} \arrow[d, "\delta_{n-1}"] \\
\mathsf{H}_{\leq n-2} \arrow[r, equal] & \mathsf{H}_{\leq n-2} \arrow[r, "f_{\leq n-2}"] & \mathsf{G}_{\leq n-2}
\end{tikzcd}$$

where $\bar{i}_1$ is the standard inclusion into the fibered coproduct. The morphism ρ is given in the diagram

$$\begin{tikzcd}
 & \mathsf{G}_n \arrow[rr, bend left, "q_{\delta_{n+1}}"] \arrow[dr, "i_2"] & & \mathsf{G}_n/\delta_{n+1} \arrow[dr, "\bar{i}_2"] \arrow[drr, bend left] & & \\
\mathsf{H}_n \arrow[ur, "f_n"] \arrow[dr, "\partial_n"'] & & \mathsf{H}_{n-1} \times^{\mathsf{H}_n} \mathsf{G}_n \arrow[rr] & & \mathsf{H}_{n-1} \times^{\mathsf{H}_n} \mathsf{G}_n/\delta_{n+1} \arrow[r, dashed, "\rho"] & \mathsf{G}_n - 1 \\
 & \mathsf{H}_{n-1} \arrow[ur, "i_1"'] \arrow[rr, equal, bend right] & & \mathsf{H}_{n-1} \arrow[ur, "\bar{i}_1"'] \arrow[urr, bend right] & &
\end{tikzcd}$$

by the universal properties of fibered coproduct and defined by

$$\rho\left([x,[y]]\right) = f_{n-1}(x)\overline{\delta_n}([y])$$

This factorization clearly factors f in degrees $k \neq n-1$. For $n-1$, we have

$$\begin{aligned}
\rho\left(\bar{i}_1(x)\right) &= \rho\left([x,[1]]\right) \\
&= f_{n-1}(x)\overline{\delta_n}([1]) \\
&= f_{n-1}(x)
\end{aligned}$$

Everything is clearly commutative except the center rectangle. The top left square is commutative by definition of the fibered coproduct. The commutativity of the top right follows from the defintion of ρ in the diagram B.2. The commutativity of the bottom left follows from the definition of $\overline{\partial_k \circ \pi_1}$ in proposition 5.3.6 and for $x \in \mathsf{H}_{n-1}$ we have

$$\begin{aligned}
\overline{\partial_k \circ \pi_1}\left(\bar{i}_1(x)\right) &= \overline{\partial_k \circ \pi_1}\left([x,[1]]\right) \\
&= \partial_k \circ \pi_1\left((x,[1])\right) \\
&= \partial_k\left(\pi_1\left([x,[1]]\right)\right) \\
&= \partial_k\left(x\right)
\end{aligned}$$

Lastly, the commutativity of the bottom right follows from

$$\begin{aligned}
\delta_{n-1}\left(\rho\left([x,[y]]\right)\right) &= \delta_{n-1}\left(f_{n-1}(x)\overline{\delta_n}([y])\right) \\
&= \delta_{n-1}\left(f_{n-1}(x)\delta_n(y)\right) \\
&= \delta_{n-1}\left(f_{n-1}(x)\right)\delta_{n-1}\left(\delta_n(y)\right) \\
&= \delta_{n-1}\left(f_{n-1}(x)\right) \\
&= f_{n-2}\left(\partial_{n-1}(x)\right) \\
&= f_{n-2}\left(\partial_{n-1}\left(\pi_1((x,[y]))\right)\right) \\
&= f_{n-2}\left(\partial_{n-1} \circ \pi_1((x,[y]))\right) \\
&= f_{n-2}\left(\overline{\partial_{n-1} \circ \pi_1}([x,[y]])\right)
\end{aligned}$$

We only have left to check that $\bar{i}_1$ and ρ are 1_{H_1}-invariant and f_1-invariant, respectively. For $a \in \mathsf{H}_1$ we have

$$\begin{aligned}
\bar{i}_1\left(x^a\right) &= [x^a,[1]] \\
&= \left[x^a,[1]^{f_1(a)}\right] \\
&= [x,[1]]^a
\end{aligned}$$

$$
\begin{aligned}
&= [x,[1]]^{1_{\mathsf{H}_1}(a)} \\
&= \bar{i}_1(x)^{1_{\mathsf{H}_1}(a)}
\end{aligned}
$$

and

$$
\begin{aligned}
\rho([x,[y]]^a) &= \rho\left(\left[x^a,[y]^{f_1(a)}\right]\right) \\
&= f_{n-1}(x^a)\,\delta_n\left([y]^{f_1(a)}\right) \\
&= f_{n-1}(x)^{f_1(a)}\delta_n([y])^{f_1(a)} \\
&= (f_{n-1}(x)\delta_n([y]))^{f_1(a)} \\
&= \rho([x,[y]])^{f_1(a)}
\end{aligned}
$$

Hence, $\left[\mathsf{H}^{f_n} : \partial^{f_n}\right]$ factors f. $\square$

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BIOGRAPHICAL SKETCH

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