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On the Geometry of Hurwitz Surfaces

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ON THE GEOMETRY OF HURWITZ SURFACES

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ABSTRACT

A Riemann surface of genus g has at most $84(g-1)$ automorphisms. A Hurwitz surface is one for which this maximum is attained; the corresponding group of automorphisms is called a Hurwitz group. By uniformization, the surface admits a hyperbolic structure wherein the automorphisms act by isometry. Such isometries descend from the (2,3,7) triangle group T acting on the universal cover $\mathbb{H}^2$.

We develop a combinatorial approach which leads to a classification of the conjugacy classes of hyperbolic elements of T, arranged by length. This allows us to study the closed geodesics of Hurwitz surfaces by performing calculations in the corresponding Hurwitz groups.

We identify the systoles and other short curves on most of the Hurwitz surfaces of genus less than 10,000. We also determine which of these surfaces are chiral and which are amphichiral. In addition, we show that certain families of closed geodesics are simple on every Hurwitz surface.

CHAPTER 1

OVERVIEW OF HURWITZ GROUPS AND SURFACES

The number of automorphisms of a Riemann surface of genus $g \geq 2$ is at most $84(g-1)$, the bound being attained only in certain special cases. This well-known and wonderful theorem is due to Hurwitz [12]. The purpose of this chapter is to explain some of the geometry and group theory behind the theorem, and to survey what is known about the associated surfaces and groups.

1.1 Hyperbolic Geometry and the Hurwitz Theorem

The idea of an automorphism, or conformal self-map, of a Riemann surface comes primarily from the theory of equations and the theory of complex functions; a few connections must be made in order to translate it into the more concrete, and possibly more intuitive, setting of hyperbolic geometry.

Topologically, a Riemann surface S of the type we wish to consider is just a closed 2-manifold of genus $g \geq 2$; its universal cover is a topological plane, which we take to be the hyperbolic plane $\mathbb{H}^2$. Now the covering of S by $\mathbb{H}^2$, according to Riemann's uniformization theorem, can be realized by a map that is conformal; furthermore, this map is unique up to conformal self-maps of $\mathbb{H}^2$. But the conformal self-maps of $\mathbb{H}^2$ are precisely the orientation-preserving isometries of $\mathbb{H}^2$. Hence, the constant-curvature geometric structure of $\mathbb{H}^2$ can be carried over to S in a way that is natural and conformally unique. It is now clear that the original automorphisms of S correspond to isometries of the new geometric structure on S; the automorphisms of course form a group, and this group is now recast as the group G of orientation-preserving hyperbolic isometries of S. It is in this sense that the hyperbolic setting may be considered more concrete, and we will generally take this point of view in the discussion that follows. (The question of orientation-reversing isometries is relevant, and will be taken up later.)

Because the group G is finite, the orbits of points of S allow the surface to be decomposed into finitely many congruent copies of a fundamental region, which will be a polygon provided G is large enough to really be interesting. This gives S the combinatorial structure of a tiling, on which G acts by permuting the faces, the edges, and the vertices. In particular, the action of G on the faces is free and transitive, so the number of faces is precisely $\#G$. Since the area of S (resulting from the hyperbolic metric) is $4\pi(g-1)$, the area A of each copy of the fundamental region is $4\pi(g-1)/\#G$, and so $\#G = 4\pi(g-1)/A$. Hence if a positive lower bound can be found for A, we will automatically have an upper bound for $\#G$ (modulo the dependence on g).

In order to discover a bound for the area A, we may lift the polygonal tiling of S to produce a tiling of the universal cover $\mathbb{H}^2$. If we can determine the area of the smallest polygon capable of tiling $\mathbb{H}^2$, this will surely be a lower bound for A. The key to this, it turns out, is to consider triangles in which the size of each angle is an integral divisor of π. Any such triangle gives a natural tiling of $\mathbb{H}^2$, generated by repeated reflections in its edges. Since the area of a hyperbolic triangle is the difference between π and its angle-sum, we simply need to find positive integers p, q and r for which the quantity

$$A = \pi - (\pi/p + \pi/q + \pi/r)$$

takes on the minimum positive value. This is equivalent to maximizing the quantity

$$\Sigma = 1/p + 1/q + 1/r,$$

subject to the constraint $\Sigma < 1$. The solution $(2, 3, 7)$ is easily found; it gives the values $\Sigma = 41/42$ and $A = \pi/42$. Since reflection in the side of a triangle reverses orientation, we take the union of two adjacent copies of the minimal triangle in order to obtain the smallest tiling polygon for which all tiling symmetries are orientation-preserving. This gives the desired result:

$$\#G \leq 4\pi(g-1)/2A = 84(g-1).$$

The discussion so far does not show whether this bound is actually attained. The fact that it is attained can best be appreciated from a group theoretic point of view, which we discuss next.

1.2 Group Theory and the Hurwitz Theorem

In addition to finding the bound on $\#G$, Hurwitz also understood the conditions that G must meet in order for the bound to be attained. These conditions derive from the structure of the tiling of $\mathbb{H}^2$ by copies of the triangle with angles $\pi/2$, $\pi/3$ and $\pi/7$. We now take a closer look at this structure.

We use the symbol Δ to refer generically to a hyperbolic triangle in which the vertex angles $\pi/2$, $\pi/3$ and $\pi/7$ occur in clockwise order around the perimeter. The mirror image of such a triangle, with corresponding angles occurring in counter-clockwise order, will be denoted by $\bar{\Delta}$.

Around a $\pi/7$ vertex the tiling consists of seven copies of Δ alternating with seven copies of $\bar{\Delta}$. Clearly the tiling has an order-7 rotational symmetry centered at this vertex. Similar statements apply to the $\pi/3$ and $\pi/2$ vertices, with rotational symmetries of order 3 and 2, respectively. It is quite easy to see that repeated rotations about the vertices of any single Δ are sufficient to generate all the copies of Δ throughout the entire tiling. These transformations of $\mathbb{H}^2$ thus comprise the $2, 3, 7$ triangle-group T of orientation-preserving symmetries of the tiling. It is a subgroup of index 2 in the group $T^{\pm}$ of symmetries generated by reflections.

Any one of the three generating rotations can be produced by composition of the other two; this makes it easy to write simple two-generator presentations of T. The most traditional one seems to be

$$T = \langle a, b \mid a^2, b^3, (ab)^7 \rangle,$$

where a and b represent, respectively, clockwise rotations of $2\pi/2$ and $2\pi/3$ about the corresponding vertices of some fixed Δ; and their product ab, interpreted as b followed by a, amounts to a counter-clockwise rotation of $2\pi/7$ about the other vertex of Δ.

Let us point out here a few easy but useful facts about the elements of T. First, there are no parabolic elements, since a parabolic isometry fixes no point of $\mathbb{H}^2$ but moves entire horoballs arbitrarily small amounts, and this is clearly incompatible with the tiling structure. Next, the only elliptic elements are the finite-order rotations about the vertices of the tiling, and these fall into conjugacy classes based solely on the (signed) angle of rotation. Hence all remaining elements are hyperbolic, or translations along geodesics, and each of these has infinite order.

Now suppose N is a proper normal subgroup of T which has finite index; let S be the quotient surface $\mathbb{H}^2/N$, and G the quotient group T/N. It follows that S is compact. It also follows that N is torsion-free; for if N contained any rotation, then it would contain a non-trivial power of a, b, or ab, hence it would contain a, b, or ab, which would then map to the identity of the quotient G. But, as can easily be seen from the above presentation for T, G would then be trivial and so N would not be proper in T. Now, since N is torsion-free, $\mathbb{H}^2$ is an *unbranched* cover of S and N is the fundamental group $\pi_1(S)$. Clearly S inherits both the constant-curvature geometry and the tiling structure of $\mathbb{H}^2$, and the number of copies of Δ (and of $\bar{\Delta}$) on S is just $\#G = [T : N]$, since $\Delta \cup \bar{\Delta}$ is a fundamental domain for T. This gives S an area of $\#G(\pi/21)$, which makes its genus $g = 1 + \#G/84$, and so we have $\#G = 84(g - 1)$. Because T is the group of (orientation-preserving) tiling symmetries of $\mathbb{H}^2$ and N is normal in T, it follows that $T/N = G$ is the group of tiling symmetries of $\mathbb{H}^2/N = S$. Thus, we have precisely the situation described by the Hurwitz theorem.

It is now clear that *any* non-trivial finite homomorphic image G of T leads to the situation just described, where we take N to be the kernel. Which groups can appear in the role of G? Based on the presentation above, a necessary and sufficient condition is that G can be generated by two elements having orders 2 and 3, respectively, and whose product has order 7. This is the characterization given by Hurwitz, and it leads (after many years, as we shall see) to most of the progress in the understanding of Hurwitz groups.

1.3 Klein's Quartic: The First Hurwitz Surface

Hurwitz knew that his bound was attained, because a surface of genus 3 with 168 automorphisms had already been studied in much detail by Klein [13, 14]. Since this surface can be defined by a fourth degree equation in complex homogeneous coordinates, it is often referred to as the Klein quartic curve.

Klein was studying a family of surfaces which occur as quotients of the $(2, 3, \infty)$ tiling of $\mathbb{H}^2$. The symmetry group of this tiling is

$$\Gamma(1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

which acts on the upper half-plane model of $\mathbb{H}^2$ by the mappings

$$z \mapsto \frac{az + b}{cz + d}.$$

For each integer $n \geq 2$ the congruence subgroup $\Gamma(n)$ consists of the elements of $\Gamma(1)$ having the form

$$\begin{pmatrix} 1 + an & bn \\ cn & 1 + dn \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}.$$

One can easily check that $\Gamma(n)$ is normal and has finite index in $\Gamma(1)$, so the quotient surface $S(n) = \mathbb{H}^2/\Gamma(n)$, symmetrically tiled by copies of the fundamental $(2, 3, \infty)$

triangle, has finite area and has as its symmetry group the quotient $\Gamma(1)/\Gamma(n)$. Of course $S(n)$ has cusps resulting from the ideal vertices of the tiling but these can easily be removed. The ideal point of each cusp acts as a center of rotational symmetry of order n; replacing each $(2, 3, \infty)$ triangle by a $(2, 3, n)$ triangle thus gives a compact surface $C(n)$ with constant-curvature geometry (hyperbolic if $n \geq 7$) while preserving the combinatorial structure, hence also the isometry group. (This is slightly untrue, since $C(n)$ may acquire additional symmetry. For example, $C(6)$ is a torus, with a continuous family of translations. But it is true enough for our current purpose.)

Klein investigated $C(1)$ through $C(6)$, which are comparatively mundane, although $C(5)$, the icosahedron, was of particular interest to him. With $C(7)$ the situation changes, and Klein was able to make explicit much of the richness of this surface. His point of view, however, was *not* that of hyperbolic geometry, which we are using here; rather, he conceived of $C(7)$ as a Riemann surface and as an algebraic curve. Indeed, Klein himself wrote later that he did not fully grasp the hyperbolic vision until much later, through the work of Poincare.

We mention here several specific points of understanding achieved by Klein:

- He understood $\Gamma(1)/\Gamma(7)$ as $PSL(2, 7)$, the simple group of order 168, and for each of its elements worked out the corresponding analytic transformation of $C(7)$.

- He explained the symmetry points in terms of their algebro-geometric roles: the 24 points where the $\pi/7$ vertices meet are the inflection points; the 56 points where the $\pi/3$ vertices meet are the contact points of bitangents; and the 84 points where the $\pi/2$ vertices meet are the sextactic points (where some conic makes order-six contact with the curve).

- He found the equation $x^3y + y^3z + z^3x = 0$ for the surface, and computed the associated vector space of abelian differentials.

- He identified the 28 symmetry lines of the surface's orientation-reversing isometries, and used them to give a model in real space, roughly a tri-axial or octahedral cone, which displays much of the combinatorial structure of $C(7)$.

1.4 Beyond Genus Three

Following the work of Klein and Hurwitz only a little progress was made over several decades. There was eventually an explosion of group-theoretic understanding, due largely to Macbeath. In this section we briefly survey the major developments.

1.4.1 Fricke's surface of genus 7

The smallest Hurwitz surface, after Klein's genus 3 quartic, turns out to have genus 7. Some writers have attributed this to Macbeath, who worked out many of its details, but Macbeath himself [18] attributes its discovery to Fricke.

The surface has symmetry group $PSL(2,8)$, the simple group of order 504. Like the genus 3 surface, it also admits orientation-reversing symmetries.

To produce an equation of the surface, Macbeath [17] first sets

$$P(z) = A_0A_1A_2A_4, \text{ where } A_i = \zeta^i z - 1 \text{ and } \zeta = e^{2\pi i/7};$$

the equation then has the form

$$y = \sqrt{P(z)} + \sqrt{P(\zeta z)} + \sqrt{P(\zeta^2 z)},$$

which can be rationalized to give a polynomial of degree 8 in y and degree 16 in z.

Macbeath also finds equations for the surface in complex projective space of dimension six, analyzes the action of $PSL(2,8)$ on homology, and finds explicit expressions for all 504 automorphisms.

1.4.2 Simple groups and extensions

As we have seen, the first two Hurwitz groups are the simple groups $PSL(2,7)$ and $PSL(2,8)$. It turns out that many other simple groups belong to the Hurwitz family. In fact, if G is a non-simple Hurwitz group, then factoring out a maximal normal subgroup will yield a simple group which is obviously a non-trivial quotient of the $(2, 3, 7)$ triangle group (since G is), hence a Hurwitz group. This suggests a search strategy: first, using the readily available classifications and data tables, determine which simple groups are Hurwitz; second, analyze their possible extensions to discover the non-simple Hurwitz groups.

In this way Macbeath [16] proved the existence of infinitely many Hurwitz groups. Specifically, he found the following two results:

- $PSL(2, q)$, where $q = p^m$, p prime, is a Hurwitz group if and only if either $q = 7$, or $q = p \equiv \pm 1 \pmod 7$, or $q = p^3$ with $p \equiv \pm 2, \pm 3 \pmod 7$.
- If G is a Hurwitz group of order $84(g-1)$, then for each positive integer n there is a group $G(n)$ of order $84(g-1)n^{2g}$ that is also a Hurwitz group. The group $G(n)$ is an extension of a product of $2g$ copies of the finite cyclic group $\mathbb{Z}_n$ by G.

There happens to be an isomorphism $PSL(2, 7) \cong PSL(3, 2)$. Cohen [3] proved that this is the only Hurwitz group of the form $PSL(3, q)$.

The alternating groups A_n are probably the most familiar non-abelian simple groups. According to results of Conder [4], almost all A_n are Hurwitz groups, beginning with A_{15}. In fact, just 64 of them are *not* Hurwitz, and these all occur in the range $1 \leq n \leq 167$.

Conder [5] also examined all simple groups of order less than one million, and showed that just 14 of them are Hurwitz. In addition, he determined the number of distinct Hurwitz surfaces corresponding to each group. His results are shown in Table 1.1 (which is essentially Table I of Conder's paper).

Another family of simple groups, the Ree groups ${}^2G_2(3^p)$, were studied by Sah [20]. He proved that they are all Hurwitz.

Table 1.1. Simple Hurwitz groups of order less than 10^6

Group	Genus	Number of surfaces
PSL(2,7)	3	1
PSL(2, 8)	7	1
PSL(2, 13)	14	3
PSL(2, 27)	118	1
PSL(2, 29)	146	3
PSL(2, 41)	411	3
PSL(2, 43)	474	3
J1	2091	7
PSL(2, 71)	2131	3
PSL(2, 83)	3404	3
PSL(2, 97)	5433	3
J2	7201	5
PSL(2, 113)	8589	3
PSL(2, 125)	11626	1

Finally, we mention a result proved by Larsen [15] using group-theoretic methods. He showed that the set of all genus values that actually occur for Hurwitz surfaces has, in an analytic sense, the same density among positive integers as does the set of perfect cubes.

CHAPTER 2

GEOMETRY OF HURWITZ SURFACES

While much progress has been made in the understanding of Hurwitz *groups*, it seems that, with the exception of Klein's own work, relatively little effort has been applied directly to questions about the geometry of Hurwitz *surfaces.*

We have seen in the previous chapter that every Hurwitz surface admits a tiling which is the image of the tiling of the universal cover $\mathbb{H}^2$ by $(2,3,7)$ triangles. This fact allows us to investigate some geometric aspects of the quotient surfaces by studying the symmetries of the universal tiling.

Definition 1 *By* $T^{\pm}$ *we mean the full* $(2,3,7)$ *triangle group of* $\mathbb{H}^2$ *isometries generated by reflections; and by* T *the index-two subgroup of orientation-preserving isometries.*

2.1 Classifying the elements of T

It is convenient to change slightly our way of looking at the (2,3,7) tiling. The 14 triangles which meet at each $\pi/7$ vertex together form a regular heptagon, with angles of $2\pi/3$ and sides formed by two copies of the shortest leg of the basic triangle. Since every triangle belongs to exactly one such heptagon, the collection of all these heptagons, meeting three at a vertex, also yields a tiling, often called the $\{7,3\}$ tiling; the symmetry groups of this new tiling are still $T^{\pm}$ and T. We use the terms *faces, edges,* and *vertices* to refer to the (topologically closed) component parts of this new

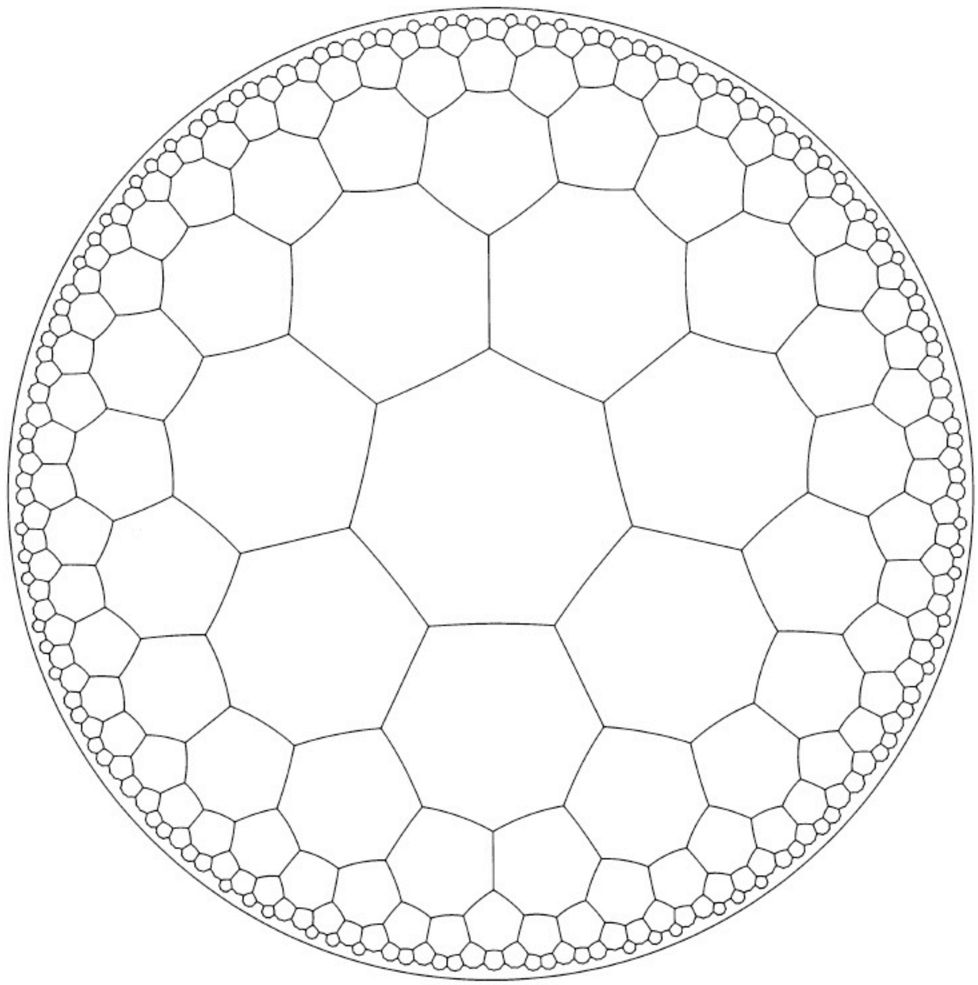

Figure 2.1. The $\{7, 3\}$ tiling of $\mathbb{H}^2$

tiling structure, whether on $\mathbb{H}^2$ or on any of the derived quotient surfaces. In most instances we refer to the midpoints of edges simply as *midpoints*. The tiling is shown in Figure 2.1, using the disk model of $\mathbb{H}^2$.

From a certain qualitative point of view, T now seems rather easy to comprehend. It maps any given face to any face in exactly seven ways. As a result, it also maps any vertex to any vertex in three ways, and any edge to any edge in two ways. In a sense this is a complete description of T, but there is much richness hidden beneath its superficial simplicity. This will become apparent as we examine the various types of elements of T.

2.1.1 Elliptics and parabolics

Every rotation, or elliptic isometry, of $\mathbb{H}^2$ has a fixed center point. If $r \in T$ is a rotation, the center point of r belongs to some face. Clearly the only possibilities for such points are the face-centers, the midpoints, and the vertices, and the tiling does in fact admit rotations with these centers. The following facts should now be obvious.

Proposition 2.1 *Each face-center, midpoint, and vertex is the fixed point of a cyclic subgroup of rotations of order 7, 2, and 3, respectively. Together these account for every rotation in T. Furthermore, any two of these rotations are conjugate if and only if they have the same signed angle of rotation.*

A parabolic isometry fixes no point of $\mathbb{H}^2$, but for each $\epsilon > 0$ moves all the points of some horoball a distance less than ϵ. Clearly there is a lower bound on the distance that a non-fixed vertex is moved by an element of T, namely the length of a single edge. Since every horoball contains vertices, we deduce the following statement.

Proposition 2.2 *No element of T is parabolic.*

2.1.2 Translations and axes

All the remaining elements of T are hyperbolic translations. Each translation is associated with a specific geodesic line, its *axis*, and a specific distance, the translation *length*, directed along that axis. The fact that in $\mathbb{H}^2$ geodesics do not occur in families of equidistant parallels makes the study of translations in T especially interesting.

Each translation $t \in T$ is conjugate to many others; in particular, we have the conjugacy class

$$[t] = \{gtg^{-1} \mid g \in T\}.$$

If X is the axis of t, then gX is the axis of gtg^{-1} and we have the class

$$[X] = \{gX \mid g \in T\}$$

.

The set of all translations having a common axis X form, along with the identity, a subgroup H_X of T. Since T does not contain arbitrarily short translations (by consideration of the distance a non-fixed vertex must move), it is evident that H_X is infinite cyclic. The generators of any such H_X will be referred to as *primitive* translations.

Proposition 2.3 *For each axis X, $[X]$ is infinite; for each translation $t \in T$, $[t]$ is infinite.*

Although this seems rather obvious geometrically, a proof will be given. We see here a dramatic difference between hyperbolic and euclidean geometry. In the checkerboard tiling of the euclidean plane, for example, each translation t belongs to a class $[t]$ having just four members.

Proof : Suppose X is the axis of some translation $t \in T$. Let F be a face which intersects X. Let c be the center of F, and $r \in T$ the rotation of $2\pi/7$ about c. Since c is quite close to X, X and rX must intersect at some point p. The translation rtr^{-1} has axis rX, moves p along rX to p_1, and moves X to an axis X_1 which contains p_1 and meets rX at the same angle as does X. If X and X_1 had a point of intersection, then X, X_1, and rX would form a triangle having angle sum greater than 2π, which is impossible. Hence X and X_1 are disjoint. By iteration we obtain a sequence of disjoint axes $X_n = rt^n r^{-1} X$. This shows that $[X]$ is infinite. Because translations having different axes are themselves different, it follows that $[t]$ is also infinite. ■

It will eventually become clear that the set $\{[t] \mid t \text{ is primitive}\}$ is also infinite.

We would like to understand the translations of T in greater detail; in particular, we would like an approach that gives geometric insight. Since conjugate translations 'look the same' geometrically, it makes sense to study translations by conjugacy class rather than individually. This is particularly appropriate in view of our interest in Hurwitz surfaces, which always correspond to quotients by translation subgroups which are *normal*, that is, unions of entire conjugacy classes.

In order to appreciate some of the complexity of T, the following exercise is useful. Let any two edges of the tiling be chosen, with a direction assigned arbitrarily to each; call these directed edges e_1 and e_Ω. There is a unique element of T which carries e_1 to e_Ω while respecting their assigned directions. Is this element a translation or a rotation? Where is its center point or translation axis? Unless the chosen edges are very close together, such isometries can be quite hard to visualize. The contrast with euclidean isometries is striking.

2.1.3 RL labels and 12 codes

We develop a solution to the exercise just given, and show how it leads to a very useful classification of all the translations in T. The ideas are quite close to Dehn's solutions to the word problem [9] and the conjugacy problem [8] for surface groups. (Stillwell's English translations [10] are helpful. Important modern developments of Dehn's ideas are found in the work of Cannon (see [1] in particular), Gromov, Thurston, and many others.)

Suppose we have an *edge-path* E in the tiling, that is, a sequence of one or more directed edges which meet head-to-tail in the natural way. Suppose further that E has no *spur*, that is, a subsequence consisting of a directed edge followed immediately by the same edge directed oppositely. We assign to E its *label*, $\lambda(E)$, which is a sequence consisting of the symbols R and L, where each occurrence of R or L corresponds to a *right* or *left* turn (with respect to some fixed orientation–intuitively, our view of

$\mathbb{H}^2$ from 'above') in the tiling at the transition between successive edges of E. Since every vertex is trivalent and E is spur-free, this rule for assigning labels makes sense.

The length of $\lambda(E)$ is one less than the length of E, with the empty label corresponding to a singleton edge-path. It is helpful to think of the route traced out by the edge-path as a sequence of steps running not from vertex to vertex, but rather from midpoint to midpoint, and making the appropriate right or left turn while passing through the interposed vertex. We emphasize the obvious but important fact that the label represents the shape of E without regard to its location in the tiling. More precisely, the same label is assigned to two edge-paths if and only if they are identified under the action of some element of T.

Given any edge-path E, there is a unique element $g_E \in T$ which maps e_1, the first directed edge, to e_Ω, the last, and we want to associate $\lambda(E)$ with g_E. But for each $h \in T$ there is a congruent edge-path hE somewhere in the tiling which also has label $\lambda(E)$, and the group element corresponding to hE is hg_Eh^{-1}. Thus $\lambda(E)$ is naturally associated not with the specific isometry g_E but rather with the entire conjugacy class $[g_E]$.

Additionally, it is easy to see that cyclic permutation of a label $\lambda(E)$ does not change the associated conjugacy class. For any E, let E' be the edge-path consisting of the subsequence e_2 through e_Ω, followed by $g_E(e_2)$. Clearly $g_{E'} = g_E$, and $\lambda(E')$ is just the first cyclic permutation of $\lambda(E)$.

By similar reasoning it is clear that the double of a label, that is, the concatenation of two copies of $\lambda(E)$, corresponds to $(g_E)^2$. The obvious generalization holds for higher powers.

We now return to the exercise suggested earlier, and give a hands-on procedure for solving it which is extremely simple, at least in principle.

Let e_1 and e_Ω be the chosen directed edges. Let E be *any* edge-path having these as its first and last elements, respectively. The element g_E, which maps e_1 to e_Ω (and is determined by them, regardless of the intervening edges of E) is the one we seek to understand. Beginning at e_1, we trace out the extended edge-path determined by multiple repetitions of $\lambda(E)$.

If g_E is a rotation, then this path will lead back to e_1 after 2, 3, or 7 repetitions. In this case, by using the symmetry of the extended path, one can gradually move center-ward in equivariant fashion until the unique center point is located.

On the other hand, if the extended path does not return within seven repetitions of $\lambda(E)$, then g_E must be a hyperbolic translation. The extended edge path converges to the attracting ideal point at one end of the translation axis; by extending in the opposite direction, interpreting $\lambda(E)$ in reverse, the path leads to the other ideal endpoint. The axis of g_E is now determined by these two endpoints. This completes our hands-on solution to the exercise.

By analyzing the combinatorial structure of labels, it is possible to systematize the study of classes of translations in T without having to manually trace out paths in a diagram. This can be done algorithmically, and yields geometric results for Hurwitz surfaces. We begin with a fact about closed paths in the tiling.

Lemma 1 *Suppose E is a spur-free edge-path of length at least two with first element e_1 and last element e_Ω. If $e_1 = e_\Omega$, then $\lambda(E)$ contains RRR or LLL as a consecutive subsequence.*

Proof : Let F be a face having edge e_1, as shown in Figure 2.2. Let D_1 be the topological disk consisting of F, and let B_1 be the set of edges forming the boundary of D_1. Generally, let D_{n+1} be the topological disk consisting of D_n along with all its neighboring faces, and let B_{n+1} be the set of edges forming the boundary of D_{n+1}. (The construction, conveniently deformed, is shown up through B_3.) Clearly there is a maximum k such that E contains an edge belonging to B_k. If $k = 1$, then since E is spur-free, the conclusion is immediate. If $k > 1$, then E contains some radial edge e_r connecting B_{k-1} to B_k, and this must be followed immediately by at least

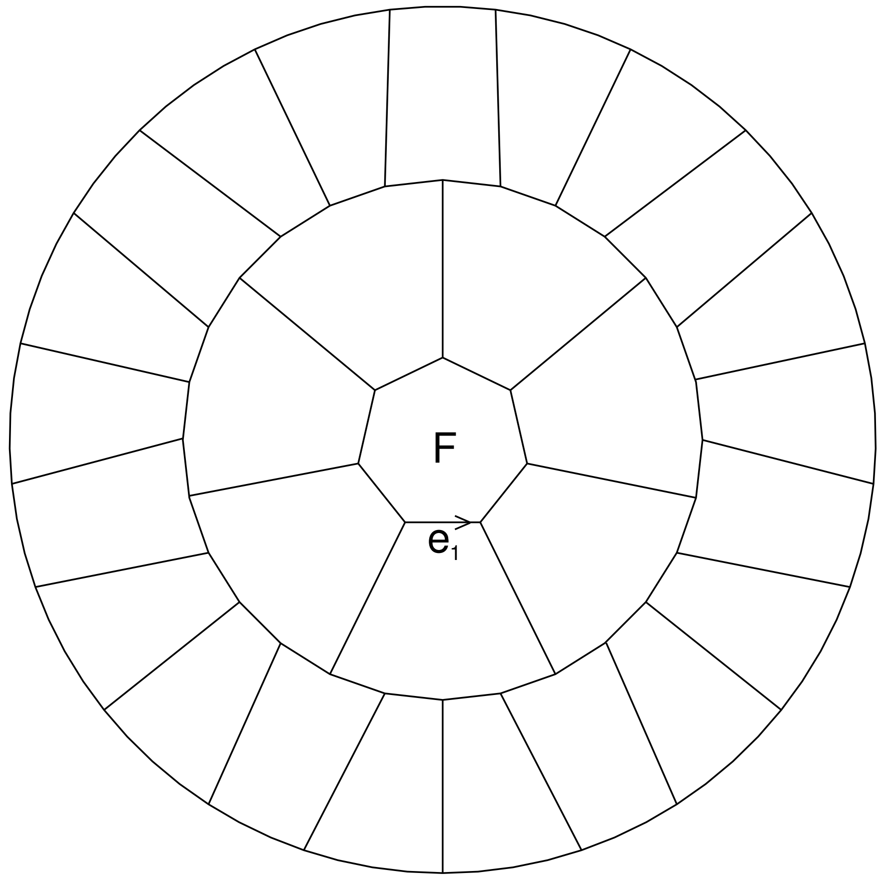

Figure 2.2. Three levels of B_n

three edges belonging to the same face as e_r. The portion of $\lambda(E)$ corresponding to this subsequence is either RRR or LLL. ■

Question 1 *How many edges belong to B_n?*

The next result follows easily from Lemma 1.

Lemma 2 *Suppose E is a spur-free edge-path of length at least two for which $\lambda(E)$, when read with cyclic repetition, contains neither RRR nor LLL. Then g_E is a hyperbolic translation.*

One might wish that the converse were also true, but of course it is not. For any hyperbolic translation $t \in T$ and any directed edge e_1, there are arbitrarily long and wandering edge-paths from e_1 to $e_\Omega = t(e_1)$, and these may contain any conceivable combination of R's and L's. Fortunately, however, it is possible to avoid such complications.

Definition 2 *Suppose $t \in T$ is a hyperbolic translation. The* label-length *of t is the minimal length taken by the label $\lambda(E)$ as E ranges over all spur-free edge-paths running from e_1 to $e_\Omega = t(e_1)$, where e_1 ranges over all edges in the tiling.*

Let us pause to consider a few simple examples. It is obvious that the empty label corresponds to the identity transformation, and that the labels R, RR, L, and LL correspond to rotations. Hence the shortest labels corresponding to hyperbolic translations are RL and LR. These differ by a cyclic permutation, so they correspond to the same set of translations. This is illustrated in Figure 2.3, where A, B, C, and D are midpoints. Because B is a center of order-two rotational symmetry, the unique geodesic segments connecting A to B and B to C together form the unique geodesic segment connecting A to C. This segment extends similarly to D, and so on, forming a geodesic line X. There is a unique element $t \in T$ which maps face F_1 to F_2 with point A going to C. Since t carries X to itself, t must be a translation with axis X. Edge vw is mapped by t to xy, so edge-path vw, wx, xy and label RL correspond to t. Edge wx is mapped to yz, so edge-path wx, xy, yz and label LR correspond to t. Edge pq maps to rs, so label RLL also corresponds to t. Obviously t has label-length 2.

We now continue the combinatorial analysis of labels.

Definition 3 *A label is* proper *if it begins with R, ends with L, and contains neither RRR nor LLL as a consecutive subsequence.*

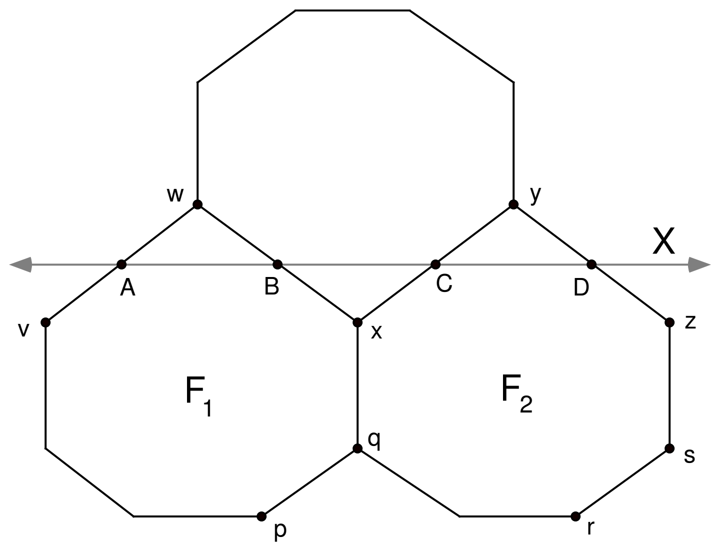

Figure 2.3. Axis of the RL translation

In other words, a proper label starts with exactly one or two R's, continues with exactly one or two L's, and so on in alternating fashion. It eventually ends with exactly one or two L's. By Lemma 2, every proper label must correspond to a translation. The converse also is true.

Lemma 3 *Every translation $t \in T$ admits a proper label.*

Proof : Suppose t is a hyperbolic translation with label-length k. Then there exists a corresponding label λ with length k. Now, λ must contain at least one R and at least one L; otherwise, it would correspond to a rotation. We may assume that λ begins with R and ends with L, since this can always be achieved by cyclic permutation.

Suppose now, for the sake of argument, that λ is not proper. Hence it contains RRR or LLL as a consecutive subsequence. We treat only the case of RRR, but the same reasoning applies for LLL. We denote by R*R a maximal string of consecutive R's in λ. By assumption, its length is at least three. Let k_R be the length of the initial string of R's in λ, and let k_L be the length of the terminal string of L's. The argument now splits into two cases, depending on the value of $k_R + k_L$.

The first case is where $k_R + k_L < k$. This implies that λ (possibly after cyclic permutation) contains the consecutive subsequence LR*RL. This can immediately be replaced by a shorter subsequence to obtain a new, shorter label which still corresponds to t. For example, LRRRL can be replaced by RLLR; this amounts to shortening an edge-path by pushing a portion of it across a face. Similarly, LRRRRL can be replaced by RLR, LRRRRRL by RR, and LRRRRRRL by the empty sequence. Additionally, any occurrence of RRRRRRR can simply be deleted. All possible instances are thus covered. However, this shortening contradicts our choice of λ, showing that this case never occurs.

The other case is where $k_R + k_L = k$. If $k_L \geq 2$, then cyclic permutation produces LR*RL and shortening occurs justs as in the first case. If $k_L = 1$, then λ is just R*RL. By tracing out edge-paths (Figure 2.1 is convenient for this), it is easy to see that RRRL labels a rotation of order 7; RRRRL, a rotation of order 3; RRRRRL, a rotation of order 2; and RRRRRRL, a rotation of order 3. Hence these do not occur. If R*R has length at least 7, then shortening occurs by deleting RRRRRRR and again we have a contradiction. So this case also cannot occur.

Hence λ is in fact proper. ■

We have shown that when a label λ corresponding to a hyperbolic translation t has length equal to the label-length of t, then λ is proper. However, the situation is complicated somewhat by the fact that not every proper label achieves this minimal length. Recall, for instance, Figure 2.3, where proper labels RL and RLL correspond to the same translation.

A more serious concern is the fact that a translation may have a number of proper labels of minimal length which are not the same, not even up to cyclic permutation. In order to be able, eventually, to compute a canonical label for each translation, we must develop some control over these sets of alternative labels.

This involves several steps, beginning with a couple of facts from hyperbolic geometry. The first one is simple and well known. The second is somewhat technical, but not too difficult to verify.

Proposition 2.4 *Any three distinct points in* $\mathbb{H}^2$ *lie on a unique circle, horocycle, equidistant curve, or geodesic.*

Proposition 2.5 *Let* C *be an equidistant curve at constant distance* d *from a geodesic* X*, with points* P*,* Q*, and* R *lying, in that order, on* C*. Suppose* Y *is a geodesic at distances* d_P*,* d_Q*, and* d_R *from points* P*,* Q*, and* R*, respectively. If* $d_Q > d$*, then* $d_P > d_Q$ *or* $d_R > d_Q$*.*

Definition 4 *Suppose* X *is the axis of a translation* $t \in T$*. The* corridor $C(X)$ *of* X *is the collection of faces in the tiling which have non-empty intersection with* X*.*

Lemma 4 *Suppose* E *is an edge-path corresponding to a translation* g_E *with axis* X*. If* $\lambda(E)$ *is proper, then* E *is contained in* $C(X)$*.*

Proof : Assume $\lambda(E)$ is proper, and let v be a vertex of E having maximal distance from X. We examine the four edges of E surrounding vertex v, that is, the two that precede it and the two that follow it. (If necessary, we allow $\lambda(E)$ to repeat cyclically in order to specify these edges.) The possible combinations of right and left turns at v and its two neighboring vertices produce just two essentially different cases, as shown in Figure 2.4. In each case we focus on v and the two nearby vertices u and w.

In Case I, vertices u, v, and w lie on an equidistant curve whose corresponding geodesic Y runs through the midpoints of the four edges. This is shown in Figure 2.5. The distance d from v to Y is quite small, clearly being less than half the length of an edge. The distance from v to X can be no greater than d; otherwise, by Proposition 2.5, the distance from X to u or w would be greater still, contrary to our choice of v.

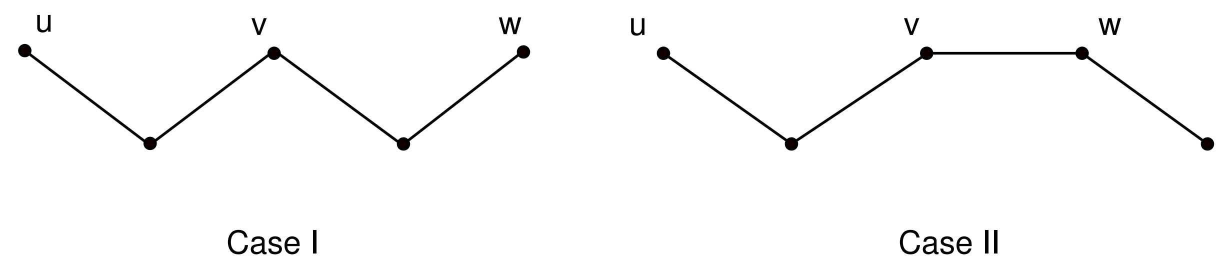

Figure 2.4. Maximally distant vertex v

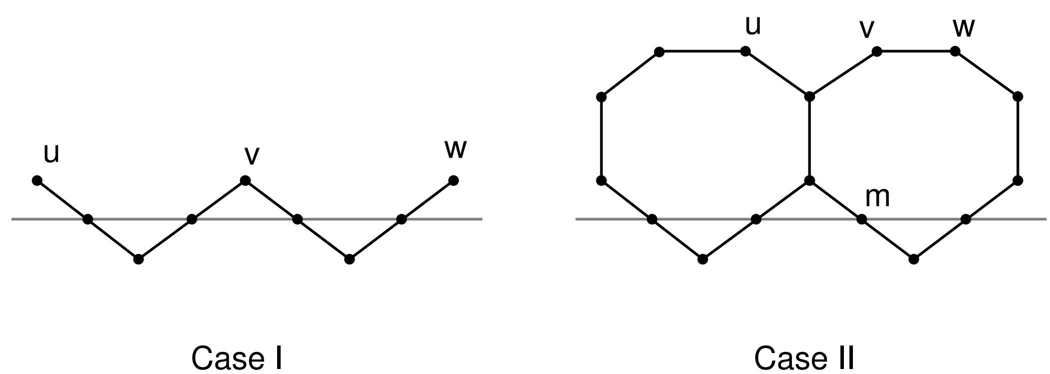

Figure 2.5. Geodesic equidistant from u, v, and w

In Case II, vertices u, v, and w also lie on an equidistant curve whose corresponding geodesic Y runs through midpoints, but now the midpoints belong to edges which are a bit farther away, as illustrated in Figure 2.5. This time the distance d from v to Y is a bit less than the distance D from v to m. Again, the distance from v to X can be no greater than d, by the same reasoning used in Case I.

Thus in both cases the distance from X to each vertex of E is less than D.

However, if E were to contain an edge not belonging to $C(X)$, then there would be an entire face not belonging to $C(X)$ contained in the region between E and X. Such a face has a vertex at a distance from X not less than the width W of a face, that is, the distance from m to w in Figure 2.5. Hence E would contain a vertex having distance at least W from X. But this is impossible, since $W > D$. Therefore every edge of E belongs to $C(X)$. ■

Since the width of a corridor is bounded, both the number of faces and the number of edges contained in it are roughly proportional to the length of the axis. Since an edge can occur at most once in a proper edge-path, it follows that the number of proper edge-paths corresponding to a given translation is, at most, roughly proportional to the length of the translation. From any one of these proper edge-paths all the others can be obtained algorithmically by a combinatorial manipulation that amounts to pushing portions of an edge-path across faces of the corridor.

It also follows that the number of edges in a proper edge-path is roughly proportional to the length of the translation. Hence there are only finitely many translations having length less than any given constant, and there is a bound on the number of proper labels that must be examined in order to find all such translations.

In practice, we find it useful to switch from RL labels to a slightly different representation. The *code* corresponding to a proper label λ is a sequence of 1's and 2's, each entry indicating the number of R's or L's occuring at successive positions in λ. For instance, the code (1 1 2 2) represents the proper label RLRRLL.

The length of a code is necessarily even, since a proper label must begin with R and end with L. The sum of the entries in any code is equal to the length of the corresponding label.

We use codes to specify a unique representative, called the *reduced code*, for each translation t, which may be determined as follows. Consider all the proper labels corresponding to t. (There are finitely many, all lying in the corridor of the axis of t.) Eliminate those which do not have minimal length. Convert the remaining labels to codes, and select the one which comes first in the the short-lex ordering. (This means that shorter codes precede longer ones, and codes of equal length are ordered lexicographically. (Everyone implicitly knows this ordering, since we use it all the time to compare integers.)) This choice is convenient algorithmically. Working with codes comes to seem quite natural, and in fact labels need not be used at all in the process of generating lists of translations.

There is one additional modification to the above definition of reduced code, and this requires some new terminology.

Definition 5 *For a translation $t \in T$, the class $[t^*]$ is the set of translations conjugate (in $T^{\pm}$) to t by an orientation-reversing element of $T^{\pm}$. An element of $[t^*]$ is called a* reflection *of t, and referred to generically by t^*.*

It is easy to see that $[t^*]$ is a conjugacy class in T (though not necessarily in $T^{\pm}$), and that its elements have the same length as t. If λ is a label for t, then a corresponding label for t^* is obtained by interchanging R's and L's in λ. The elements of $[t^{-1}]$ and $[(t^*)^{-1}]$ likewise have the same length as t. A label for $(t^*)^{-1}$ is obtained simply by reversing λ; applying the R-L interchange to this reversed label then yields a label for t^{-1}.

We now complete the definition of reduced code by specifying that the minimum be selected not merely from among the codes for t, but in fact from the codes for t, t^{-1}, t^*, and $(t^*)^{-1}$. Thus a reduced code represents $[t] \cup [t^{-1}] \cup [t^*] \cup [(t^*)^{-1}]$ rather than just $[t]$. This may seem like a complication, but in fact it tends to simplify the associated algorithms.

2.1.4 Axes with special symmetries

Every hyperbolic translation $t \in T$ has an axis X, and the way that X is situated in the tiling has consequences for the geometry of Hurwitz surfaces. For a primitive translation t, one can imagine traveling along X while observing the local landscape of the tiling. After traversing a distance equal to the length of t, the scenery must repeat. Every axis has this type of periodicity, which is simply a manifestation of the infinite cyclic subgroup generated by t.

Some axes have more than the basic periodic symmetry. These higher symmetries are of five different types. One possible approach to this analysis would be to study the subgroups which preserve the axis in question. We take a somewhat more qualitative approach, since our interest has little to do directly with these stabilizing subgroups. As usual, we are more concerned with the classes of T-equivalent axes than with individual axes.

We define each type of symmetry by giving a geometric characterization. Discussion and examples follow.

Definition 6 *An axis has* type A *symmetry if it is the axis of a glide-reflection in $T^{\pm}$. An axis has* type D *symmetry if it is fixed setwise, with exactly one fixed point, by a reflection in $T^{\pm}$. An axis has* type S *symmetry if it is fixed setwise by an order-two rotation in T. An axis has* type ADS *symmetry if the characterizations of types A, D, and S all apply. An axis has* type I *symmetry if it is the mirror-line of a reflection in $T^{\pm}$.*

The names chosen for these types are intended to be symbolic mnenomic aids. The arrow-like shape of the symbol A suggests motion in the direction of its vertical symmetry axis, so it seems natural to associate it with glide-reflections. The upper-case D has a perpendicular line of reflection symmetry, as do the axes of type D. The letter S has order-two rotational symmetry, and the symbol I has several symmetries resembling those of the type I axes.

The diagram in Figure 2.6 illustrates each of the five types with a schematic 'axis' having the approprite symmetry. It is a worthwhile exercise to locate the mirror-lines and rotational centers for each axis.

Of all the symmetries, type I is the easiest to recognize, since it pertains to just one class of axes. Any edge in the tiling is a geodesic segment, and can therefore be extended to an entire geodesic line. As the segment extends, it successively bisects a face, then an edge, then a second face; finally it coincides with another edge,

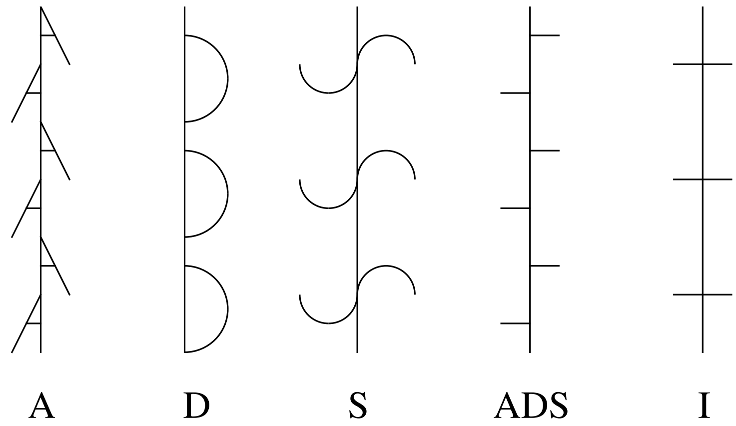

Figure 2.6. The five symmetry types

and the pattern continues. This repetition makes it clear that the line is in fact a translation axis. It is also the mirror-line of a reflection symmetry in $T^{\pm}$. In fact, every mirror-line of the tiling belongs to this same class of axes, since a mirror-line must pass through the interior of some face, thus bisecting it, and there is essentially only one way to bisect a face. Hence the mirror-line is an axis of type I, and every axis of type I is fundamentally the same. We call these axes I-lines.

Besides the translation and mirror-line symmetries, an I-line also admits order-two rotational symmetries centered at the midpoints of the bisected and the coincident edges. Hence it has type S symmetry. It is orthogonal to the I-lines which intersect it at these midpoints, so it is fixed setwise by the corresponding reflections. Hence it has type D symmetry. Since it is a mirror-line and admits translations, it also admits glide-reflections. Hence it has type A symmetry. The I-lines are thus of type ADS. The reduced code for a primitive I-line translation is (1 2 2 2).

Question 2 *Apart from the I-lines, does any axis contain a vertex or a face-center?*

The next easiest symmetry to understand is probably type S. Since a type S axis admits an order-two rotation, it must contain the rotational center, and this can only be a midpoint. Now let X be the axis, p the midpoint it contains, r the order-two rotation about p, and t a primitive translation along X. Let p' be the image of p under t, and q the point on X halfway between p and p'. The composition $tr \in T$ preserves X setwise while fixing only q; hence tr is an order-two rotation centered at q, and therefore q is also a midpoint. Hence X contains rotational centers regularly spaced at intervals of half the length of t.

Conversely, let p and q be any two midpoints in the tiling, X the unique geodesic containing them, and d the distance from p to q. The composition of successive order-two rotations at p and q preserves X setwise, while translating it a distance $2d$. Hence X is a translation axis, and has type S symmetry. It now follows from the preceding paragraph that every type S axis can be obtained in this way.

It is not hard to see that each axis of type S admits a label which is *anti-palindromic*; this means that reading the label backward is the same as reading it forward but with all R's and L's interchanged. For instance, RLLRRL is anti-palindromic (see Figure 2.7). It corresponds to the shortest translation which has symmetry only of type S. Another short example is given by the reduced code (1 1 2 1 2 2). This translation has no anti-palindromic proper label; its symmetry manifests only through improper labels, such as RRLLLRRRLL. This shows that some care is needed in designing algorithms which detect symmetry types.

Type D symmetry pertains to any axis preserved by reflection in a perpendicular mirror-line; but the only mirror-lines of reflections in $T^{\pm}$ are the I-lines. Hence a type D axis is simply one that meets an I-line at a right angle. It is not immediately clear whether there exist many classes of such axes; however, it is easy to see that such an axis must admit palindromic labels. For example, the proper label RLRLL permutes cyclically to give LRLRL, which is palindromic and thus corresponds to a translation having an axis of type D.

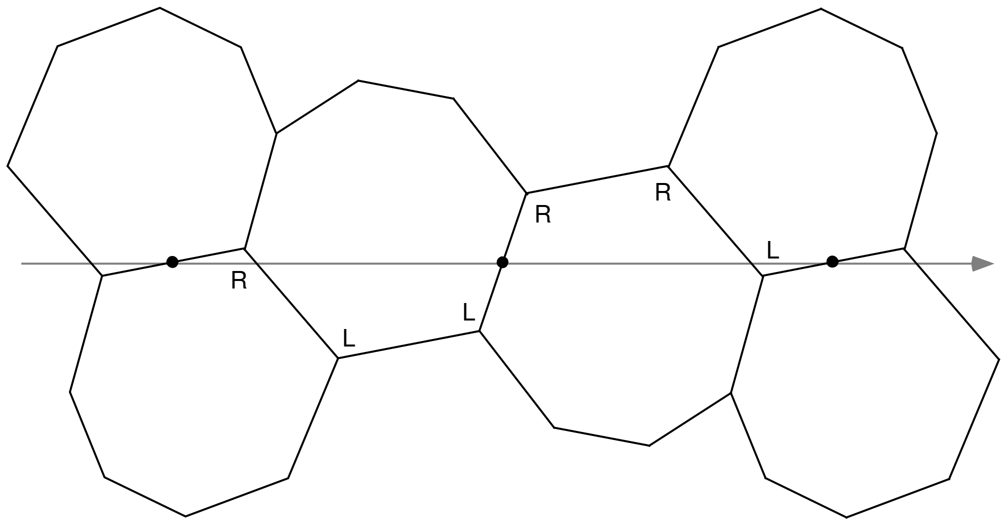

Figure 2.7. Axis of the RLLRRL translation

The code for this translation is (1 1 1 2). Generalizing this to (1 1 1 1 1 2), (1 1 1 1 1 1 1 2), and so on, gives an infinite sequence of distinct primitive translations, all having palindromic labels. Hence there are infinitely many classes of type D axes.

Let X be an axis of type D, p the intersection of X with a perpendicular I-line M, and t a primitive translation along X. The image of M under t is another I-line perpendicular to X; hence there is an infinite family of copies of M which meet X perpendicularly at regularly spaced intervals. Now let $m \in T^{\pm}$ be the reflection in M, p' the image of p under t, and q the point on X halfway between p and p'. The composition $tm \in T^{\pm}$ preserves X setwise, fixing only q, and is orientation-reversing; hence tm is a reflection in an I-line meeting X at q, and so we have a second family of perpendicular I-lines alternating with those of the first family.

Of all the symmetries, type A seems hardest to visualize. Although many of the shorter translations have axes of type A, they invariably have D and S symmetries as well. Perhaps this is what makes it difficult to get a good intuitive grasp of the distinct geometric character of type A symmetry. The shortest translation having symmetry of type A but not D or S is given by the code (1 1 1 1 2 1 1 2 2 1 1 1 1 2 1 1 2 2).

It has already been mentioned that I-lines have type A symmetry, but in fact this is somewhat misleading. The primitive glide-reflection of an I-line has the same length as its primitive translation, whereas all other axes of type A, since they are not mirror-lines, have glide-reflections that are half the length of their primitive translations.

Therefore, such a translation has a label that is an *anti-double*; this means it is the concatenation of two copies of an RL sequence, but with the R's and L's interchanged in the second copy. The label corresponding to the above-mentioned code (1 1 1 1 2 1 1 2 2 1 1 1 1 2 1 1 2 2) is an example of this. It is now easy to find a generalization of this code which yields an infinite sequence of distinct primitive translations. Hence there are ininitely many classes of type A axes.

As already mentioned, I-lines have symmetry types A, D, and S. Many other axes of type ADS are found for relatively short translations. Just as for types A and D, it can readily be shown using labels that there are infinitely many classes of type ADS axes. One infinite family, for instance, is given by (1 1 2 1 1 2), (1 1 1 1 2 1 1 1 1 2), (1 1 1 1 1 1 2 1 1 1 1 1 1 2), etc.

Theorem 2.1 *If an axis has any two of the symmetry types A, D, and S, then it has the third as well.*

Proof : This is not difficult, so we give an argument for just one case. Suppose X has symmetry types D and S; we must show it also has type A. Let p be a midpoint on X, and r the corresponding order-two rotation. There is a family of mirror-lines orthogonal to X; let M be one of these which is nearest to p, and let m be the reflection in M. Let d be the distance from p to M. If $d = 0$, then rm is a reflection fixing X pointwise; hence X is an I-line, and we are finished. If $d > 0$, then rm is a glide-reflection (of length $2d$) along the axis X; hence X has type A symmetry, as claimed. ■

Our characterizations of the symmetry types as given in Definition 6 are closely tied to the geometry of the tiling. They can be restated, however, in a succinct group-theoretic way without directly mentioning the underlying geometry. As a bonus, Theorem 2.1 follows immediately from this new characterization.

Theorem 2.2 *Suppose $t \in T$ is a translation with axis X. Then X has type A symmetry if and only if $[t] = [t^*]$; X has type D symmetry if and only if $[t^{-1}] = [t^*]$; and X has type S symmetry if and only if $[t] = [t^{-1}]$.*

Those familiar with the classification of planar symmetries will recognize that our five symmetry types represent five of the seven so-called frieze groups. Of the two remaining frieze groups, one corresponds to the basic translation symmetry (which we have not distinguished as a type, since every axis has it), and the other is easily seen not to occur in the present context, since the corresponding axis would have to be a mirror-line admitting no order-two rotation. This shows that our classification into types is complete.

2.1.5 The length spectrum of T

Each hyperbolic translation $t \in T$ has an axis X, and a length $\ell(t)$ which is simply the distance from p to $t(p)$ for any point $p \in X$. (If $p \notin X$ the translation distance is strictly greater then $\ell(t)$.) It follows immediately that $\ell(t^{-1}) = \ell(t)$, and is geometrically obvious that $\ell(gtg^{-1}) = \ell(t)\ \forall g \in T^{\pm}$. Hence, as was pointed out earlier, all elements of $[t]$, $[t^{-1}]$, and $[t^*]$ have the same length.

Definition 7 *For each translation $t \in T$ the* similarity class *of t, denoted by $[[t]]$, is defined by $[[t]] = [t] \cup [t^{-1}] \cup [t^*] \cup [(t^*)^{-1}]$.*

Definition 8 *The* length spectrum *of T, denoted by $\mathcal{L}_T$, is the non-decreasing ordered set of all positive numbers which occur as lengths of hyperbolic translations $t \in T$; the multiplicity of each element $\ell \in \mathcal{L}_T$ is the number of distinct similarity classes whose elements have length ℓ.*

We have so far avoided any explicit numerical representation of elements of T. In order to calculate lengths, we will now make use of the well-known representation of $\text{Isom}^+(\mathbb{H}^2)$ given by

$$PSL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\} \text{ mod } \pm I,$$

where matrices act on the upper half-plane by the linear fractional transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \left(\frac{az + b}{cz + d} \right).$$

There is one particularly useful formula that we shall need.

Lemma 5 *In the upper half-plane model of $\mathbb{H}^2$, rotation by 2θ about the point yi on the imaginary axis is given by*

$$\begin{pmatrix} \cos\theta & y\sin\theta \\ -\frac{1}{y}\sin\theta & \cos\theta \end{pmatrix}.$$

Proof : Let r be the given matrix. A simple calculation verifies that $r(yi) = yi$. Hence r, having a fixed point, must be a rotation. Next we compute the derivative

$$\frac{d}{dz} r(z) = \left(-\frac{z}{y} \sin\theta + \cos\theta \right)^{-2};$$

evaluating at $z = yi$ gives

$$\begin{aligned}(-i\sin\theta + \cos\theta)^{-2} &= (\cos^2\theta - \sin^2\theta - 2i\cos\theta\sin\theta)^{-1} \\ &= (\cos 2\theta - i\sin 2\theta)^{-1} \\ &= \cos 2\theta + i\sin 2\theta.\end{aligned}$$

Since our model of $\mathbb{H}^2$ is conformal, this shows that the rotation angle of r is 2θ. ■

We now position the tiling so that a vertex v is located at i, and the center of an adjacent face F is located at bi (with $b > 1$) on the imaginary axis. The hyperbolic distance between these points is just the length h of the hypotenuse of the fundamental (2,3,7) triangle, and so we set $b = e^h$. Basic trigonometric formulas give

$$\cosh h = (\cot \frac{\pi}{3})(\cot \frac{\pi}{7}) \approx 1.19888,$$

from which we calculate

$$b = e^h = \cosh h + \sinh h = \cosh h + \sqrt{\cosh^2 h - 1} \approx 1.86018.$$

Rotations which generate T are now given by

$$A = \begin{pmatrix} \cos \frac{\pi}{3} & \sin \frac{\pi}{3} \\ -\sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{pmatrix} \approx \begin{pmatrix} 0.5 & 0.86603 \\ -0.86603 & 0.5 \end{pmatrix},$$

which gives an order-three rotation about i, and

$$B = \begin{pmatrix} \cos \frac{\pi}{7} & b \sin \frac{\pi}{7} \\ -\frac{1}{b} \sin \frac{\pi}{7} & \cos \frac{\pi}{7} \end{pmatrix} \approx \begin{pmatrix} 0.90097 & 0.80710 \\ -0.23325 & 0.90097 \end{pmatrix},$$

which gives an order-seven rotation about bi.

Let e_1 be the edge of F running from $B^{-1}(i)$ to i. Now e_1 followed by $B(e_1)$ forms an edge-path of length two having label L. Similarly, e_1 followed by $A^{-1}B(e_1)$ forms an edge-path of length two having label R. Accordingly, we define transformations

$$R = A^{-1}B \quad \text{and} \quad L = B.$$

Hence, by simply reinterpreting any label λ as the composition of a sequence of transformations R and L, a matrix is obtained which represents an element $t \in T$ having label λ. In this way the classification of classes of translations by their labels can be used to easily calculate the associated lengths. The necessary formula is

$$\ell(t) = 2 \cosh^{-1} \left| \frac{1}{2} \text{Trace}(M) \right|,$$

where $M \in PSL(2, \mathbb{R})$ represents T.

It is now clear how to compute the length spectrum of T, or at least any finite portion of it. Because label-length and hyperbolic length are quasi-isometric quantities, one has only to work through finitely many proper labels in order to be certain of obtaining all translations up to any desired length. This ensures completeness of the computed portion of the spectrum.

The first 35 entries of the length spectrum are presented in Table 2.1, which along with the length gives the symmetry classification and the reduced code for each similarity class. Symmetry of type I is not indicated. Every translation results from some positive number of iterations of an appropriate primitive translation, and this number, which we call the *periodicity*, is given in the fourth column under the heading 'Symmetries'.

Three different repeated lengths occur in this first small portion of the spectrum. Although this multiplicity may seem surprising at first sight, it should probably be expected in light of the fact that all entries of the matrices R and L are algebraic numbers. In fact, as one examines longer portions of the spectrum, multiplicity seems to be very common.

Although we have not rigorously analyzed the precision of our length calculations, we believe that virtually all observed multiplicity is real. As evidence, we note that among the first 30,000 entries of the spectrum, calculated lengths which differ always do so within the first five decimal places; those that agree to five places in fact agree to at least fifteen places.

Another portion of the spectrum is shown in Table 2.2. (Its fifth entry is the translation mentioned on page 29, which has symmetry only of type A.) The large amount of apparent multiplicity seen in the table is typical of most portions of the spectrum that we have examined. It is only at the beginning of the spectrum that repeated lengths seem to be rare.

Question 3 *To what extent can the multiplicity really be understood?*

Table 2.1. Length spectrum of T (entries 1-35)

Length	Symmetries				Code
0.98398656	A	D	S	1	(1 1)
1.73600575	A	D	S	1	(2 2)
1.96797312	A	D	S	2	(1 1 1 1)
2.13110501		D		1	(1 1 1 2)
2.66193075			S	1	(1 1 2 2)
2.89814945		D		1	(1 2 2 2)
2.95195969	A	D	S	3	(1 1 1 1 1 1)
3.15482446		D		1	(1 1 1 1 1 2)
3.47201150	A	D	S	2	(2 2 2 2)
3.54271043	A	D	S	1	(1 1 2 1 1 2)
3.62731599			S	1	(1 1 1 1 2 2)
3.80470389			S	1	(1 1 2 1 2 2)
3.93594625	A	D	S	4	(1 1 1 1 1 1 1 1)
3.93594625		D		1	(1 1 1 2 2 2)
4.15197165		D		1	(1 1 1 1 1 1 1 2)
4.20180693	A	D	S	1	(1 2 2 1 2 2)
4.26221003		D		2	(1 1 1 2 1 1 1 2)
4.39145967			S	1	(1 1 2 2 2 2)
4.48925712			S	1	(1 1 1 1 2 1 1 2)
4.60473270			S	1	(1 1 1 1 1 1 2 2)
4.65401425		D		1	(1 2 2 2 2 2)
4.76043291			S	1	(1 1 1 1 2 1 2 2)
4.84179776				1	(1 1 1 2 1 1 2 2)
4.91993281	A	D	S	5	(1 1 1 1 1 1 1 1 1 1)
4.93876271		D		1	(1 1 1 1 1 2 2 2)
5.01321723			S	1	(1 1 1 2 1 2 2 2)
5.14067614		D		1	(1 1 1 1 1 1 1 1 1 2)
5.20801725		D		1	(1 1 1 2 2 1 2 2)
5.20801725	A	D	S	3	(2 2 2 2 2 2)
5.28890090		D		1	(1 1 1 1 1 2 1 1 1 2)
5.28890090		D		1	(1 1 2 1 1 2 2 2)
5.32386150			S	2	(1 1 2 2 1 1 2 2)
5.35145856			S	1	(1 1 1 1 2 2 2 2)
5.42679721	A	D	S	1	(1 1 1 1 2 1 1 1 1 2)
5.45942685			S	1	(1 1 1 1 1 1 2 1 1 2)

Table 2.2. Length spectrum of T (entries 1509-1542)

Length	Symmetries				Code
10.7627571				1	(1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 2 1 1 2)
10.7627571				1	(1 1 1 1 1 1 1 1 1 2 1 1 2 2 2 1 2 2)
10.7627571		D		1	(1 1 1 1 2 1 1 1 1 2 2 1 1 2 1 1 2 2)
10.7627571		D		1	(1 1 1 2 1 1 1 2 2 2 2 1 2 2 2 2)
10.7645791	A			1	(1 1 1 1 2 1 1 2 2 1 1 1 1 2 1 1 2 2)
10.7668487			S	1	(1 1 1 1 1 2 1 1 2 1 1 1 1 1 2 2 2 2)
10.7668487				1	(1 1 1 1 1 2 1 1 1 1 1 2 2 1 1 2 2 2)
10.7668487				1	(1 1 1 1 1 1 1 2 2 1 1 2 1 1 1 2 2 2)
10.7668487			S	1	(1 1 1 1 1 1 1 1 1 1 2 2 1 2 2 1 2 2)
10.7668487			S	1	(1 1 1 1 2 1 1 2 2 1 1 2 1 1 1 1 2 2)
10.7668487			S	1	(1 1 1 1 2 1 1 2 2 1 1 1 1 2 2 1 1 2)
10.7668487				1	(1 1 1 2 2 1 1 2 2 1 2 2 2 1 2 2)
10.7668487				1	(1 1 1 2 1 1 2 2 2 2 1 2 2 1 2 2)
10.7719392				1	(1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 2 1 1 1 2)
10.7719392			S	1	(1 1 1 1 1 1 2 1 1 2 2 1 1 2 2 1 1 2)
10.7719392				1	(1 1 1 1 2 1 1 1 2 1 2 2 2 2 1 1 1 2)
10.7719392				1	(1 1 1 1 1 2 1 2 2 1 2 2 2 2 2 2)
10.7719392		D		1	(1 1 1 2 2 2 1 1 1 2 2 2 1 2 2 2)
10.776012				1	(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2)
10.776012			S	1	(1 1 1 1 2 1 1 1 1 2 1 1 1 1 2 2 2 2)
10.776012				1	(1 1 1 1 1 1 1 2 1 1 2 2 1 1 1 2 2 2)
10.776012			S	1	(1 1 1 1 2 1 1 2 1 1 1 1 2 2 1 1 2 2)
10.776012			S	1	(1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2)
10.776012			S	1	(1 1 1 2 1 1 2 1 1 1 2 1 1 2 1 2 2 2)
10.776012				1	(1 1 1 1 1 1 2 1 1 2 1 2 2 1 2 2 1 2)
10.776012				1	(1 1 1 1 2 1 1 1 2 1 1 2 2 1 2 2 1 2)
10.776012				1	(1 1 1 2 2 1 2 2 1 1 2 2 1 2 2 2)
10.776012				1	(1 1 2 2 2 1 2 2 2 2 2 2 2 2)
10.7833302				1	(1 1 1 1 1 2 1 1 2 2 1 1 1 1 1 2 2 2)
10.7833302				1	(1 1 1 1 1 1 2 1 1 2 2 1 1 2 1 1 2 2)
10.7833302				1	(1 1 1 1 2 1 1 2 1 1 2 2 1 1 1 1 2 2)
10.7833302				1	(1 1 1 1 2 1 1 1 2 1 1 1 2 2 2 2 1 2)
10.7833302				1	(1 1 1 2 2 1 1 2 2 1 2 2 1 2 2 2)
10.7833302				1	(1 1 2 2 2 2 2 1 2 2 2 2 2 2)

In the euclidean world, multiplicity of translation legths also occurs. For instance, in the $\mathbb{Z}\oplus\mathbb{Z}$ checkerboard tiling, the translations $(8,1)$ and $(7,4)$ have the same length. The explanation is number-theoretic, involving the distinction between primes of form $4k+1$ and those of form $4k-1$. In T the situation, though possibly analogous, is certain to be more complicated.

2.2 Generating Hurwitz quotients

Having a good portion of the length spectrum of T now at our disposal, we show how it can be used to produce many Hurwitz surfaces. Our method is subject to various criticisms, as it relies on the computational power of modern-day technology and also employs a good deal of trial and error. It avoids nearly all of the finite group theory that has made possible the theorems of Macbeath, Conder, and others. On the other hand, it allows one not familiar with those theorems to easily produce enough examples to nurture an interest in the subject, and it seems to offer more geometric insight than do the group-theoretic results.

The method consists simply in writing down a group presentation of a certain form, and then instructing a computer to calculate the order of the group. If the calculated order is 1, then the group is trivial and no surface is obtained. If the calculated order is $n > 1$, then we have a Hurwitz group and a corresponding surface of genus g, where $n = 84(g-1)$. If the computer exhausts its resources before reaching an answer, then we know nothing; the group may be infinite, finite, or even trivial.

Since a Hurwitz group is simply a finite quotient of T, we begin with a presentation of T and add to it one or more extra relators; this is equivalent to forming the quotient of T by the normal closure (in T) of the set of extra relators. The extra relators must be translations, since factoring out a rotation corresponds to a surface having orbifold singularities. (Additionally, it is easy to see that factoring out any rotation must yield a trivial quotient, so there is actually just one such orbifold surface.)

This can all be done without knowing the length spectrum in any detail. After pondering over the tiling for a while, it becomes easy to visualize several of the shorter translations and to see how to produce them by composing the preferred rotational generators. This is all that is needed to begin producing interesting surfaces.

Here is an example illustrating most of the details. We start with the presentation

$$T = \langle a, b \mid a^3, b^7, (ab)^2 \rangle,$$

where b is the rotation by $2\pi/7$ about the center of some face F, and a is the rotation by $2\pi/3$ about the vertex located at one corner of F. The composition ab, representing b followed by a, is easily seen to be the rotation of order two about a midpoint on the perimeter of F.

The composition bba^{-1} produces a primitive translation t with label RL. We take t^4 as the extra relator, and run the computation. This is shown in the following transcript from an interactive session with the GAP computer algebra system [11], which is used for all our group calculations.

```
gap> f:=FreeGroup(2);
<free group on the generators [ f1, f2 ]>
gap> a:=f.1 ;;  b:=f.2 ;;  t:=b*b*a^-1 ;;
gap> g:= f / [ a^3, b^7, (a*b)^2, t^4 ];
<fp group on the generators [ f1, f2 ]>
gap> Size(g);
168
```

Thus the group g has order 168, and is the symmetry group of the corresponding surface of genus 3; we know this can only be the Klein surface.

The normal closure $\langle\langle t^4 \rangle\rangle$, by elementary covering space theory, is the fundamental group of the surface. It often plays a major role in the study of a surface, but here it does not and we do not need it in any more explicit form. (It is somehow quite satisfying to see it in this form which, although relatively useless, is very concise and requires no arbitrary, symmetry-breaking choices, as in the usual presentations.)

It is fascinating to consider what happens geometrically when this quotient surface is formed. Every RL axis in $\mathbb{H}^2$ simultaneously 'rolls up' to form a closed loop with length equal to the translation length of t^4, and even though these axes intersect extensively, all this rolling up takes place in a compatible way. There are exactly 21 of these closed loops in the quotient surface. Other axes also roll up to form closed loops, but their attributes are not so obvious.

If t^6 rather than t^4 is used as the extra relator then the resulting surface has genus 14. Using t^7 also gives a surface of genus 14. Although homeomorphic, these two surfaces are geometrically different; if not, the total number of symmetries would exceed the Hurwitz bound of $84(g-1)$. There is yet another surface of genus 14, which can be produced by taking the extra relator to be $(\mathrm{RRRLLL})^3$ (that is, the third power of a primitive translation with label RRRLLL). Extra relators for several other surfaces are tabulated in Appendix B.

Often, especially for surfaces of higher genus, two (or more) extra relators can be used together to produce a finite quotient, while the use of either one by itself appears not to (that is, the computer exhausts its resources). This is not at all surprising, and conceivably one could show rigorously that the apparently larger quotients really are infinite. However, there are also surfaces of relatively high genus which can be produced by using just one extra relator. This seems interesting, and can perhaps be restated most clearly in terms of the fundamental group.

Question 4 *For which Hurwitz surfaces can the fundamental group be normally generated (in T) by a single translation?*

For instance, using $(RL)^9$ and $(RLLRRLL)^6$ together yields a surface of genus 13123, and we have found no single extra relator which gives the same result. On the other hand, using $(RLRLRLLRRLL)^3$ by itself produces a surface of genus 14197.

Table 2.3. Primitive translations that generate

Symmetries				Length	Code	Genus
A	D	S	1	8.85587906	(1 1 2 1 1 2 2 1 1 2 1 1 2 2)	7
		S	1	10.21836713	(1 1 1 1 1 1 2 1 1 2 1 1 1 1 2 2 1 2)	7
	D		1	10.61581822	(1 1 1 1 1 2 1 2 2 2 2 2 2 2 2 1 2)	7
A			1	11.17826694	(1 1 1 1 2 1 1 2 1 2 2 1 1 2 2 2 1 2)	7
			1	12.41565658	(1 1 1 1 1 2 1 1 2 2 1 1 1 2 1 1 2 2 2 2)	7
		S	1	9.173078188	(1 1 1 1 1 2 1 1 2 1 1 1 2 2 1 2)	17
			1	11.8911705	(1 1 1 1 1 1 1 2 1 1 1 2 1 1 2 2 1 2 2 2)	17
A	D	S	1	10.45126176	(1 1 1 1 1 1 1 2 1 2 2 1 1 1 2 2 1 2)	118
	D		1	12.65211909	(1 1 1 1 1 1 1 1 1 2 2 1 1 2 1 1 1 2 1 1 2 2)	118
		S	1	10.87325845	(1 1 1 1 2 2 2 1 1 2 1 2 2 2 2 2)	129
A	D	S	1	12.8563103	(1 1 1 1 1 1 1 1 2 2 2 1 1 1 1 1 1 1 1 2 2 2)	129
A	D	S	1	11.44155775	(1 1 1 1 2 2 1 2 2 1 1 1 1 2 2 1 2 2)	146
A	D	S	1	11.53150714	(1 1 1 2 2 1 1 2 2 1 1 1 2 2 1 1 2 2)	146
A	D	S	1	11.87644894	(1 1 1 1 1 1 1 2 1 1 2 1 1 1 1 1 1 1 2 1 1 2)	146
A	D	S	1	12.3197837	(1 1 2 1 1 2 2 2 2 1 1 2 1 1 2 2 2 2)	146
A	D	S	1	12.43189638	(1 1 1 1 1 1 1 1 1 2 1 2 2 1 1 1 1 1 2 2 1 2)	411
A	D	S	1	12.43189638	(1 1 1 1 2 2 2 2 2 1 1 1 1 2 2 2 2 2)	411
A	D	S	1	12.67294552	(1 1 1 1 2 1 1 1 1 2 2 1 1 1 1 2 1 1 1 1 2 2)	411
A	D	S	1	11.74745067	(1 1 1 1 1 2 1 1 1 1 2 1 1 1 1 1 2 1 1 1 1 2)	474

A somewhat similar phenomenon leads to a question involving primitive translations. The genus 3 surface, for example, can be obtained by using as the sole extra relator either $(\mathrm{RL})^4$ or $(\mathrm{RLRRLRRLLLRLL})^2$ (or any of several other possibilities); but is there any *primitive* translation which achieves the same result? In fact there are several. The reduced codes (1 1 1 1 1 1 2 1 1 1 1 1 1 2), (1 1 1 1 1 2 2 1 1 1 1 1 2 2), and (1 1 1 1 1 1 1 1 2 2 2 2 2 2) provide three different solutions to the problem. These, along with the other examples shown in Table 2.3, have been found experimentally. A theoretical explanation of this would be quite pleasing.

Question 5 *For which Hurwitz surfaces can the fundamental group be normally generated (in T) by a single primitive translation?*

In addition to the $84(g-1)$ orientation-preserving symmetries, a Hurwitz surface may or may not admit additional symmetries which reverse its orientation. If it does, then there are exactly $84(g-1)$ of them and the surface is *amphichiral* (Greek for 'both-handed'). Otherwise the surface is *chiral*.

Chirality is very easy to detect in the surfaces produced by the method of factoring out extra relators. Suppose N is the normal closure of the set of extra relators. Clearly N is normal in T; but it may or may not be normal in $T^{\pm}$, and this is precisely what chirality depends on.

If each extra relator t has symmetry type A or D, then N contains both $[t]$ and $[t^*]$; hence $N \lhd T^{\pm}$ and the surface is amphichiral. On the other hand, if some extra relator(s) t_i do not have A or D symmetry, then N may or may not contain each $[t_i^*]$, and this can be checked by one additional computation. Namely, we augment the set of extra relators by including for each t_i a representative t_i^*, and compute the size of the new quotient group. If it is unchanged, then the additional relators were already in N, hence $N \lhd T^{\pm}$ and again the surface is amphichiral. On the other hand, if the new quotient is strictly smaller, then $t_i^* \notin N$ for at least one of the t_i^*; hence T is the largest subgroup of $T^{\pm}$ in which N is normal. Therefore no orientation-reversing element of $T^{\pm}$ is able to descend to a symmetry of the quotient surface, and so the surface is chiral.

The smallest chiral Hurwitz surface is the one of genus 17. One way to obtain it is by taking $(RLRRLL)^3$ as the extra relator. When the reflected translation $(LRLLRR)^3$ is included as an additional extra relator, the quotient surface has genus 3. This is what tells us that the genus 17 surface is chiral; it also indicates that the smaller surface is covered by the larger one. Additional computation indicates that while the image of $(RLRRLL)$ has order 3 in N, the image of $(LRLLRR)$ actually has order 6. This fact helps give a concrete grasp of what chirality means, and also makes clearer the nature of the covering map.

Table 2.4. Some chiral Hurwitz surfaces

Genus	Translation t	Order(t)	Order(t^*)	
17	(1 1 2 2)	3	6	Covers genus 3
2091	(1 1 2 2)	5	6	J_1
2091	(1 1 2 2)	5	10	J_1
2091	(1 1 2 2)	6	10	J_1
7201	(1 1 1 1 2 2)	5	15	J_2
7201	(1 1 2 1 2 2)	3	12	J_2
8065	(1 1 2 2 1 2 2 1 2 2)	2	4	Covers genus 3, 7, 17, and 1009
17473	(1 1 1 2 1 1 2 2)	3	6	Covers genus 3, 14, 17, and 2185

When $(RLRRLL)^4$ is used instead of $(RLRRLL)^3$, the resulting surface has genus 411 and seems to have a good chance of being chiral. However, when $(LRLLRR)^4$ is added to the mix, the same quotient results; hence the surface is amphichiral.

An assortment of chiral Hurwitz surfaces is presented in Table 2.4. For each surface, a primitive translation t is given by its reduced code, along with an integer n under the heading Order(t), indicating that t^n is used as an extra relator (and that the image of t in fact has order n in the quotient). The order of a reflected translation t^* appears in the next column; the inequality of these two orders is the manifestation of chirality. The Janko groups J_1 and J_2 appear repeatedly; since they are simple, no covering of a smaller surface is possible. Interestingly, J_1 and J_2 each have an associated amphichiral surface, as well.

Among the surfaces so far examined, chirality is rather uncommon. This may continue to be the case as the Hurwitz family continues on to higher genera, but we know of only one specific result on this point. Gareth Jones has recently informed us that each of the Ree groups (which form an infinite family) has a corresponding chiral Hurwitz surface. There may also be some significance in the fact that an amphichiral surface can have chiral covers; we have not noticed any occurrence of the reverse phenomenon.

Question 6 *How common is chirality among Hurwitz surfaces?*

2.3 Curves on Hurwitz surfaces

One way to study the geometry of a surface is by trying to understand the nature of the curves that it supports. But there are far too many curves to really deal with, so which ones do we study? In topology the trick is to lump together entire homotopy classes of closed curves, and these classes are then tractable. In geometry, especially when negative curvature is involved, it is preferable to go to the opposite extreme and pick out one curve from each non-trivial free homotopy class. Since we are treating Hurwitz surfaces in terms of their hyperbolic structure, the obvious choice is to pick the closed geodesics and from now on, unless otherwise indicated, this is what we mean when referring to curves. It is well known that in every class such a curve exists, is unique, and is shorter than every other member of the class. In studying these curves we are interested in such things as their length, number, arrangement, and intersections.

2.3.1 Systoles

A curve of minimal length is called a *systole* (SIS-t'-lee, from a Greek word meaning 'to contract' [19]). The fact that systoles exist on Hurwitz surfaces follows immediately from the discrete combinatorics underlying the length spectrum. Here we present several easy results involving the systoles of Hurwitz surfaces, some of which could surely be adapted to more general classes of surfaces. Our proofs are intentionally rather informal, since the results are probably obvious or well known to most readers.

Proposition 2.6 *Each systole of a Hurwitz surface is simple.*

Proof : If it were not simple, then it would be a union of shorter non-geodesic loops, at least one of which must be non-trivial. ■

Proposition 2.7 *A Hurwitz surface has more than one systole.*

Proof : Suppose α is a systole, and let F be a face which intersects α. Let p be the center point of F, and let r be the symmetry which rotates $2\pi/7$ about p. Clearly $r(\alpha)$ is a systole. But since α is quite close to p, it must intersect $r(\alpha)$. Hence they are distinct curves, by Proposition 2.6. ■

Proposition 2.8 *Each systole intersects another.*

Proof : This follows from the proof of Proposition 2.7. ■

Proposition 2.9 *Two systoles have at most one point of intersection.*

Proof : Suppose systoles α and β intersect at distinct points p and q. These points separate α into two arcs; let $\breve{\alpha}$ be the shorter one. Define $\breve{\beta}$ likewise. Now $\breve{\alpha}$ and $\breve{\beta}$ together form a non-geodesic loop γ with length less than or equal to a systole, so γ must be trivial. But this is impossible, since it is formed by two geodesic arcs. ■

Proposition 2.10 *Each systole is non-separating.*

Proof : Suppose α is a systole which separates. By Proposition 2.7 there is a systole β which intersects α. Geodesic intersections are necessarily transverse and β is simple, so there must be an even number of intersection points (since α separates). But this violates Proposition 2.9. ■

2.3.2 Length spectra

The lengths of closed geodesic curves form a system of numerical characteristics of a given surface. The systole is merely the shortest of these curves.

There is a close connection between the axes of T and the curves on a Hurwitz surface. Suppose X is an axis with corresponding primitive translation $t \in T$ of length $\ell(t)$, and let p be a point lying on X and belonging to the interior of some face F. Hence $t(p)$ belongs to face $t(F)$, and in particular the crossing of F by the arc $X \cap F$ is exactly congruent to the crossing of $t(F)$ by the arc $X \cap t(F)$.

Now consider a Hurwitz surface S and the covering projection π from the tiling of $\mathbb{H}^2$ to S. Clearly $\pi(X)$ is a geodesic on S crossing faces $\pi(F)$ and $\pi t(F)$, and t_*, the homomorphic image of t induced by π, is a symmetry of S mapping $\pi(F)$ to $\pi t(F)$ and preserving $\pi(X)$ setwise. But the symmetry group of S is finite. Hence the order of t_* is some finite integer n; in particular t_*^n fixes $\pi(F)$. This means that $\pi(X)$, after looping about S in its own particular way, returns and simply retraces the same path over and over. In other words, X projects to a closed geodesic with length $n \cdot \ell(t)$.

Conversely, suppose α is a closed geodesic of length $\ell(\alpha)$ on S. Then α lifts to a geodesic X in $\mathbb{H}^2$ which clearly runs through faces of the tiling in a manner that is repetitive with period $\ell(\alpha)$. Consequently, X is the axis of a translation $t \in T$ and $\ell(t) = \ell(\alpha)$. Although t is not necessarily primitive, it is of course some positive power of a primitive translation.

Definition 9 *The* length spectrum $\mathcal{L}(S)$ *of a Hurwitz surface S is the non-decreasing ordered set of lengths of closed geodesic curves on S. Each element $\ell \in \mathcal{L}(S)$ occurs with multiplicity $m(\ell)$ equal to the number of distinct curves (up to free homotopy and reversal of direction) having length ℓ.*

If S is a Hurwitz surface for which we know the associated Hurwitz group G_S in some explicit form, it then becomes straightforward to compute $\mathcal{L}(S)$ using the length spectrum $\mathcal{L}_T$. We simply take each primitive translation class $[[t]]$ in order, beginning with the shortest, and compute the order n of t_* using a known label for t. The corresponding closed geodesic of length $n \cdot \ell(t)$ is often called a *prime* geodesic;

the non-prime geodesics are those which wind two or more times around the path of some prime geodesic.

There are two additional relevant considerations. The first one has to do with chirality. If S is amphichiral, then the curves arising from $[t]$ and from $[t^*]$ all have the same length, since they are equivalent under orientation-reversing symmetries of S. Hence their length can be found with a computation using a single representative label.

However, if S is chiral, then for a translation t having neither type A nor type D symmetry the curves arising from $[t]$ and from $[t^*]$ are not equivalent under any symmetry of S and therefore may have lengths. Hence two separate computations are needed to determine the two potentially different lengths.

The second consideration is the question of how many different curves arise on S from a given class $[[t]]$. Due to the high symmetry of S we expect this number to be fairly large, but its exact value is not obvious and in fact depends on the symmetry type of t.

If S is amphichiral, then its full symmetry group (including orientation-reversing isometries) has order $168(g-1)$. Thus, for a given curve α, we should expect to find $168(g-1)$ copies of α situated symmetrically throughout S. However, α may be preserved setwise by a number of symmetries; if we can count these, then the orbit-stabilizer theorem will tell us how many geometrically distinct copies of α actually occur.

Counting the stabilizing symmetries boils down to two things: the order n of the induced symmetry t_*, where t is a primitive translation along an axis X that projects to α, and the symmetry type of X. If X has none of the five symmetry types, then the n distinct powers of t_* comprise the stabilizer of α and so there are $\frac{168(g-1)}{n}$ copies of α. If X has just one of the symmetry types A, D, or S, then there are an additional n ways of preserving α, and so there are $\frac{168(g-1)}{2n}$ curves. If X has symmetry type ADS or I, then even more symmetries preserve α, and the number of curves turns out to be $\frac{168(g-1)}{4n}$.

On the other hand if S is chiral then there are a total of $84(g-1)$ symmetries, all of them orientation-preserving. Thus an axis having no special symmetry corresponds to $\frac{84(g-1)}{n}$ separate curves; the same formula holds for an axis having symmetry only of type A or type D, since these do not descend to symmetries of a chiral surface. An axis having symmetry of type S (including those of type ADS and type I) does retain the extra symmetry following projection to S, and the number of curves is thus given by $\frac{84(g-1)}{2n}$.

Tables of calculated length spectra, including the number of curves in each class, are given for a number of different surfaces in Appendix C.

For convenience, only the prime curves are represented in the tables. Of course, one could easily calculate their multiples and include these values in the table, if desired.

2.3.3 Simple closed curves

Suppose we are given a translation $t \in T$ with axis X. In general there does not appear to be an easy way of telling whether X descends to a simple or a non-simple curve on any particular Hurwitz surface. It is therefore somewhat surprising that certain classes of axes can be shown to descend to simple curves on *every* Hurwitz surface.

We begin by stating two useful lemmas, without proof. Both are easily seen to be true by lifting to the universal cover.

Lemma 6 *If geodesic curves have an arc in common, then they coincide everywhere.*

Lemma 7 *A directed geodesic curve cannot traverse the same arc in one direction and again in the opposite direction.*

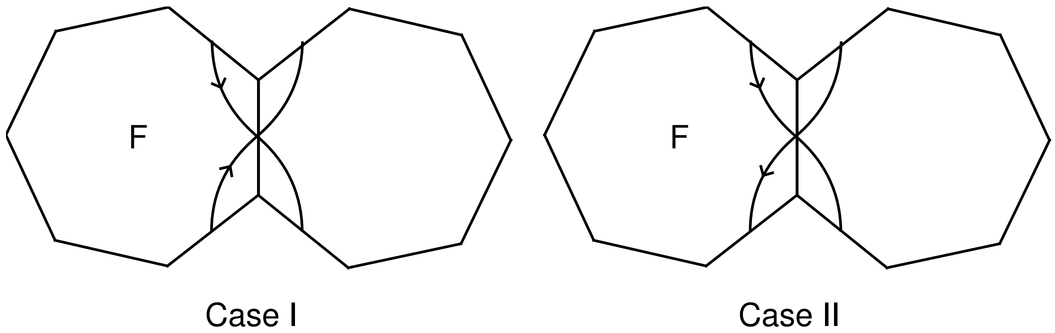

Figure 2.8. The two possible RL intersections

Theorem 2.3 *The RL curves are simple on every Hurwitz surface.*

Proof : Suppose α is a directed non-simple RL curve, and F a face which it crosses. There are essentialy just two ways in which the self-intersection of α could occur, as shown in Figure 2.8.

We treat Case I first. As shown in Figure 2.9, F contains directed arcs ab and cb, both of which belong to α. Let r be the symmetry which rotates F by $2\pi/7$, so that ab maps to bc and bc to cd. By Lemma 6 (and by keeping track of the direction of arcs) it follows that directed arc cd belongs to α. Applying r five more times shows that directed arcs ed, ef, gf, ga, and finally ba, all belong to α. But this is impossible, by Lemma 7.

For Case II, we first extend the intersecting arcs to a neighboring face G, as illustrated. Virtually the same argument used in Case I applies, except that now we take r to be the rotation of G by $4\pi/7$. Repeatedly applying r shows that directed arcs ab and ba both belong to α, again violating Lemma 7. ■

Theorem 2.4 *The RRLL curves are simple on every Hurwitz surface.*

Proof : Suppose α is a directed non-simple RRLL curve. There are essentialy just four ways in which the self-intersection of α could occur, as shown in Figure 2.10.

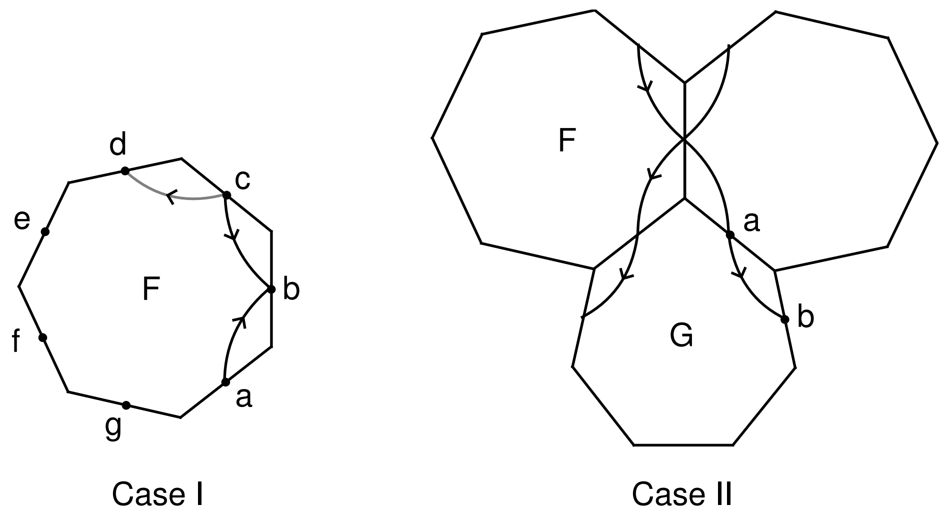

Figure 2.9. RL intersections with additional arcs

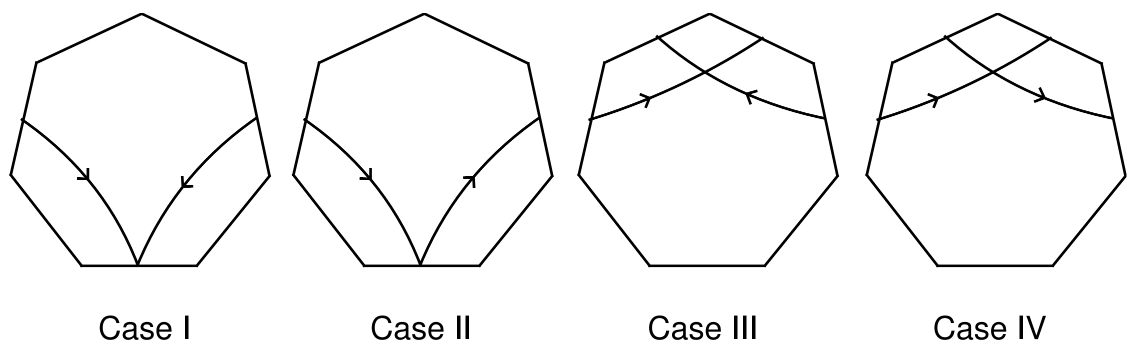

Figure 2.10. The four possible RRLL intersections

Cases I and III can be handled in the same manner as Case I of Theorem 2.3. Cases II and IV can be eliminated by a slight variation of the argument used for Case II of Theorem 2.3. ■

Theorem 2.5 *The RLRLL curves are simple on every Hurwitz surface.*

Proof : These curves also have label RRRLLL, which is improper but easier to visualize. Just as in the previous two theorems, the present type of curve crosses a

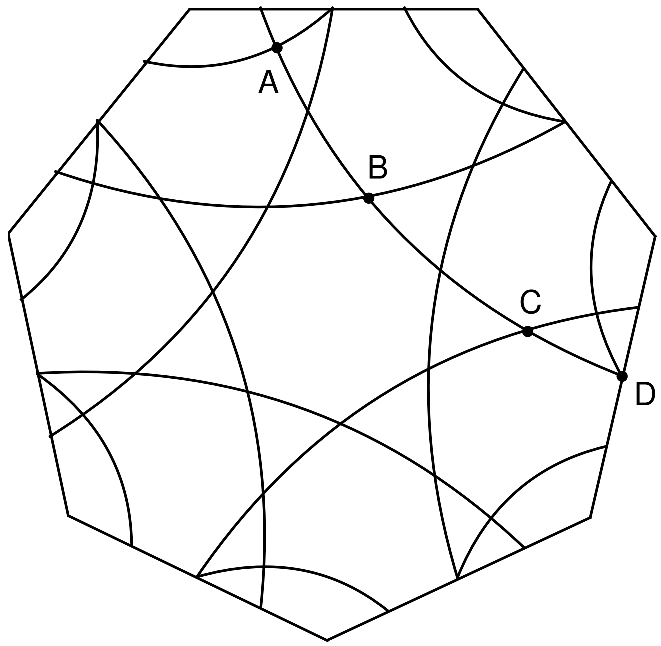

Figure 2.11. All RLRRLL crossings of a single face

face in essentially only one way. This time six different cases arise, but these are very similar to the four cases of Theorem 2.4, so we omit further details. ■

For curves that are a bit more complicated combinatorially, there is a different argument that works quite well.

Theorem 2.6 *The RLRRLL curves are simple on every Hurwitz surface.*

Proof : Suppose α is a non-simple RLRRLL curve on surface S. There are two different ways that α can cross a face, as can be seen in Figure 2.7. Consideration of all possible crossings of a single face, as shown in Figure 2.11, demonstrates that there are essentially just four types of possible self-intersection, corresponding to points A, B, C, and D. Assuming that α is directed, each type of intersection gives rise to two cases. We will discuss one case in detail; the remaining seven follow the same logic and will be described briefly.

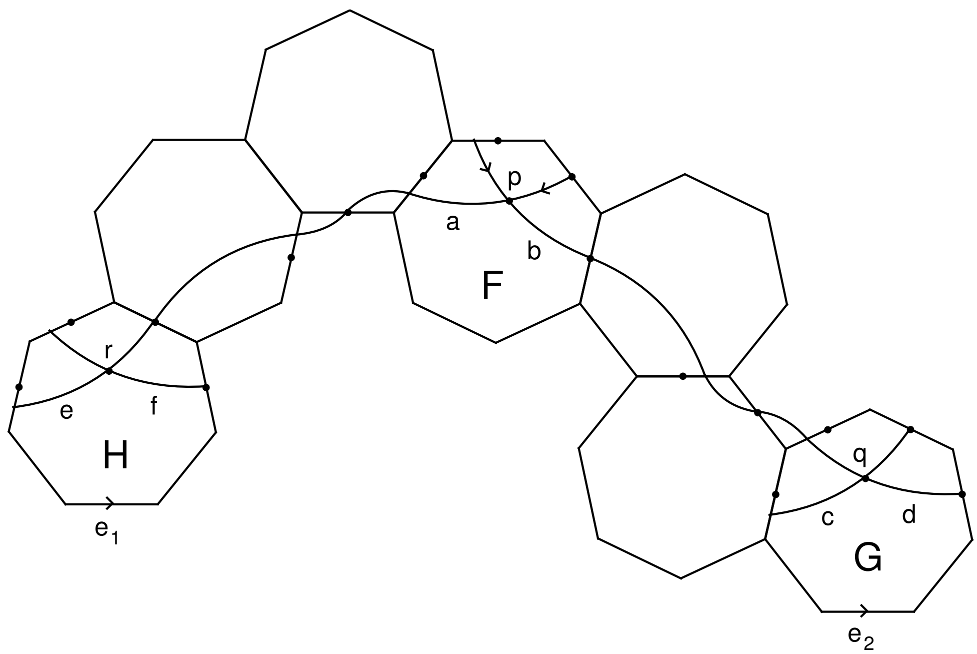

Figure 2.12. One case of RLRRLL intersection

For the first case let us take an intersection of type B, with α directed as shown in Figure 2.12; the intersection point is now labeled p (on face F), and the extension of α to nearby faces is illustrated. Arcs a and b are the intersection of α with F. A number of midpoints are highlighted for convenience in tracing out the path of α.

In the tiling of $\mathbb{H}^2$ let X be a directed axis whose image on S is α. Let $t \in T$ be the primitive translation in the forward direction along X, and t_* its image in the symmetry group of S. Thus t preserves α setwise while shifting each of its points forward a constant distance $\ell(t)$.

This means that t_* maps point p to q, arc b to d, face F to G, and arc a to c (which shows that c belongs to α, by Lemma 6). But t_* must also map p to r, a to e, F to H, and b to f (so f also belongs to α). Hence G and H are actually the same face, with matching corresponding parts. In particular, directed edges e_1 and e_2 coincide.

We now treat Figure 2.12 as a view of the universal cover of S. Portions of four different lifts of α are visible, and we already know that G and H are two lifts of the same face. Let $u \in T$ be the unique isometry that maps e_1 to e_2. By following an edge-path from e_1 to e_2, we find that u has label LLRLRRLLRLRRLL.

Since e_1 and e_2 coincide on S, u belongs to the group of covering transformations. Hence the symmetry group of S is a quotient of $T/\langle\langle u \rangle\rangle$. This latter group can be computed by simply taking the label of u as an extra relator (in the manner of Section **??**), and it turns out to be trivial. Hence the symmetry group of S is trivial; but this is impossible since S is a Hurwitz surface. This completes the argument for the first case.

The other seven cases are handled in exactly the same fashion. Different labels turn up for u, namely RLLRRLR, LRLRLLR, LRRLRLLRLRL, RLLRRLRLRL, LLRRLRLRL, LLRLRRLRRLLR, and RRRLRLRRL. In each case the quotient $T/\langle\langle u \rangle\rangle$ is trivial, thus completing the proof. ■

Theorem 2.7 *The RLLRRLL curves are simple on every Hurwitz surface.*

Proof : These curves are images of I-lines. Five of them labeled A through E are shown crossing a single face in Figure 2.13. Pairs of these arcs form the various types of possible intersection. The five pairs AB, AC, AD, DE, and AE cover all essentially different potential crossings; the first four each produce two cases when the curve's direction is taken into account, while AE produces just one. Thus there are a total of nine essentially different cases.

The six cases arising from intersection types AB, AC, and AD are easily dealt with by the technique used in Theorem 2.3. The final three cases can be handled with the method used in Theorem 2.6. The details are easy enough to verify, but we omit them in order to avoid excessive repetition. ■

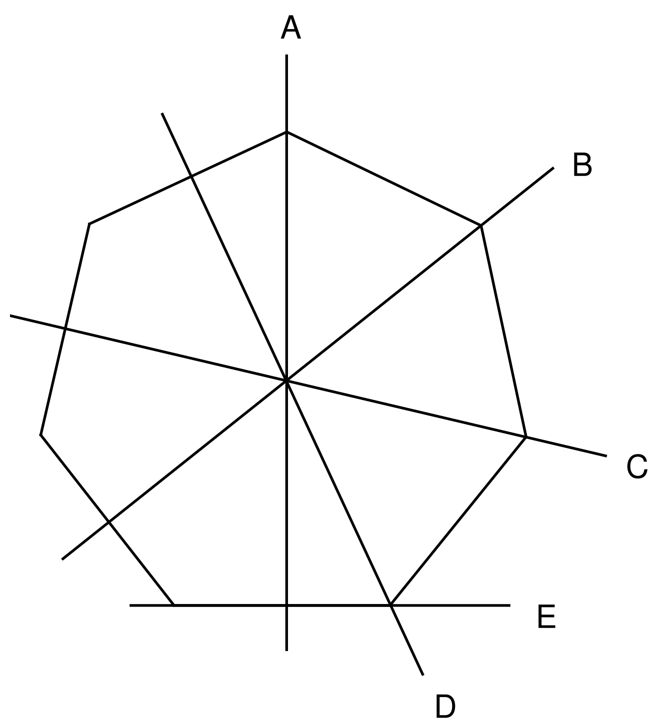

Figure 2.13. Five types of intersection of RLLRRLL curves

APPENDIX A

REMARKS ON KLEIN'S CURVE

We note here a couple of interesting observations regarding the genus 3 surface. Both can be readily seen using Klein's diagram of a fundamental domain for the surface, shown in Figure A.1. This domain is a regular 14-gon, with edge-pairings indicated by the numbers around the perimeter. It is surprising that the domain admits symmetries (namely, rotations of order 14) which do not extend to the Hurwitz surface, and both observations really arise from this fact.

The first observation has to do with a certain curve on the surface. It is clear in Figure A.1 that the domain is bisected vertically by an I-line. But there is also a horizontal bisector which is not an I-line; on the Hurwitz surface it extends geodesically and closes after passing six more times through the center point of the domain, which is also a face-center. This provides an affirmative (but very limited) answer to Question 2. The associated primitive translation is (1 1 1 1 1 1 1 2).

The second observation relates to the tiling of Klein's surface by 24 heptagons. The tiling sketched in Figure A.1 is of course the regular tiling having 168 symmetries. Now, if a rotation of order 14 is applied to the tiling (but not to the edge-pairings), then a new tiling of the *same* Hurwitz surface is obtained. It cannot be regular, since that would give the surface too many symmetries. So the tiling is non-regular but is built from the same 24 heptagons, with the same local structure, as in the Hurwitz tiling. The possibility of this happening is quite unexpected. In group-theoretic terms, this means that T has subgroups A and B (the deck transformations corresponding to the two differently-tiled versions of the surface) which are not merely

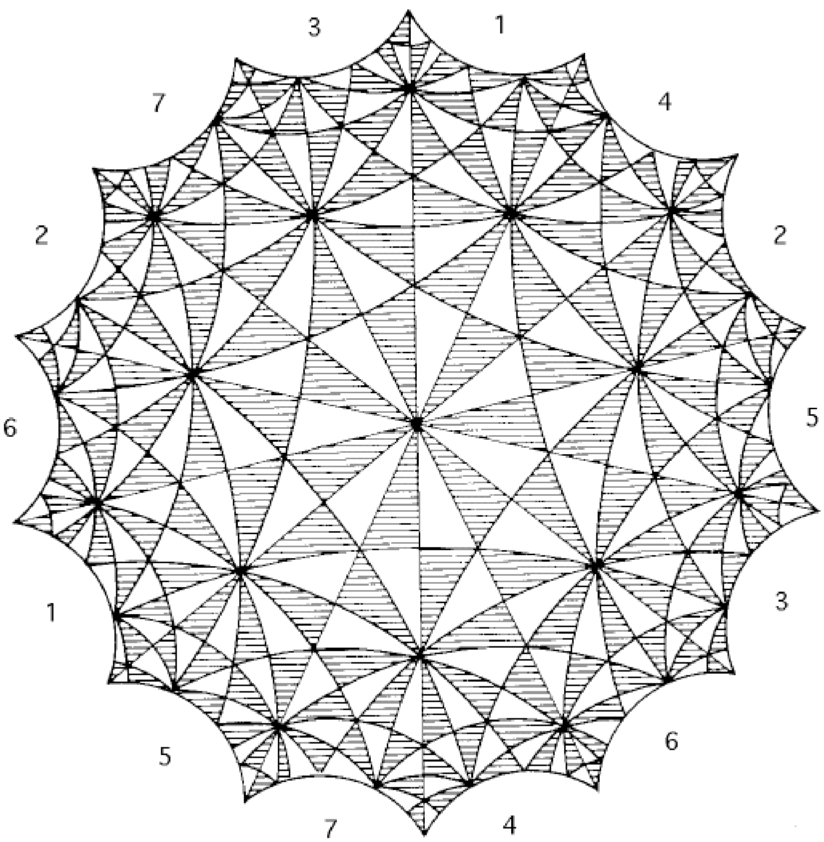

Figure A.1. Fundamental polygon for genus 3 surface

isomorphic but in fact conjugate in $\mathrm{Isom}^+(\mathbb{H}^2)$; but in T, one of them is normal and the other is not.

Question 7 *Is this phenomenon as unique as it seems, or are there other examples?*

There is an obvious geometric connection here to the multiplicities of the length spectrum, and we hope to be able to determine exactly how much of the multiplicity arises directly from the twin tilings. Algebraically, both phenomena have to do with elements that are conjugate in $\mathrm{Isom}^+(\mathbb{H}^2)$, but fundamentally different in T.

APPENDIX B

EXTRA RELATORS FOR GENERATING SURFACES

Every Hurwitz group can be obtained by adding one or more extra relators to the presentation

$$T = \langle a, b \mid a^7, b^3, (ab)^2 \rangle;$$

a natural geometric interpretation of the generators gives an explicit connection between the group and the corresponding surface (see Section 2.2 for details).

The following table gives an adequate set of extra relators for the surfaces whose Hurwitz groups are simple and of order less than one million. Where multiple surfaces occur with the same genus, they are indicated, for example, as 146A, 146B, etc. For identification of the groups, see Table 1.1 on page 9.

Our preference is for small sets of short relators, but we do not claim any sort of optimality for the ones given here.

Table B.1. Convenient extra relators

Genus	Relator(s)
3	$(1\ 1)^4$
7	$(1\ 2\ 2\ 2)^2$
14A	$(1\ 1)^6$
14B	$(1\ 1)^7$
14C	$(1\ 1\ 1\ 2)^3$
118	$(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2)^3$
146A	$(2\ 2)^5$
146B	$(1\ 1\ 1\ 2)^5$
146C	$(1\ 1\ 2\ 2)^5$
411A	$(1\ 1)^{10}$, $(1\ 2\ 2\ 2)^5$
411B	$(1\ 1\ 2\ 2)^4$
411C	$(1\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 2)^2$
474A	$(1\ 1)^{11}$, $(1\ 1\ 1\ 2\ 1\ 2\ 2\ 2)^3$
474B	$(1\ 1\ 2\ 1\ 1\ 2)^3$
474C	$(1\ 2\ 2\ 2\ 2\ 2)^3$
2091A	$(1\ 1)^{11}$, $(1\ 1\ 2\ 2)^5$
2091B	$(1\ 1\ 1\ 2)^6$, $(1\ 1\ 2\ 2)^5$
2091C	$(1\ 1)^{10}$, $(1\ 1\ 2\ 2)^6$
2091D	$(1\ 1\ 1\ 1\ 1\ 2)^5$, $(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2)^2$
2131A	$(1\ 1\ 1\ 1\ 1\ 2)^4$
2131B	$(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 1\ 2)^2$
2131C	$(1\ 1\ 1\ 2\ 2\ 1\ 1\ 1\ 2\ 2)^2$
3404A	$(1\ 1\ 2\ 2\ 2\ 1\ 1\ 2\ 2\ 2)^2$
3404B	$(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 1\ 2)^2$
3404C	$(1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2)^2$
5433A	$(1\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 2\ 2\ 2\ 2)^2$
5433B	$(1\ 1\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 2\ 1\ 1\ 2\ 2)^2$
5433C	$(1\ 1\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 2\ 2)^2$
7201A	$(1\ 1)^{10}$, $(1\ 1\ 1\ 1\ 2\ 2)^5$
7201B	$(1\ 1\ 2\ 2\ 1\ 2)^3$
7201C	$(1\ 1)^{12}$, $(1\ 1\ 2\ 1\ 1\ 2\ 2\ 2)^3$
8589A	$(1\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 2\ 2\ 2\ 2)^2$
8589B	$(1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 1\ 2\ 2\ 2\ 2)^2$
8589C	$(1\ 1\ 2\ 1\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 2\ 2)^2$
11626	$(1\ 2\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 2)^2$

APPENDIX C

TABULATED LENGTH SPECTRA

The tables presented here give portions of the spectra of prime geodesics for all the Hurwitz surfaces having simple Hurwitz groups of order less than one million. The labels 146A, 146B, etc., are the same as in Appendix B.

Several of the surfaces are chiral; this is noted in the captions. In these cases, the starred translations must be read as reflected codes. That is, the corresponding label begins with L and ends with R.

The spectra are complete for geodesics having length up to 14.49074723. Beyond this length, there are surely omissions in the tables. (Missing entries are those arising from primitive translations having label length greater than 34.) This distinction is marked by a narrow horizontal break at the appropriate spot in each table. In a few cases the break occurs right at the beginning, indicating that we have not succeeded in finding even the systoles.

We note that surface 411C has two classes of systoles. This seems to be the only known example of such behavior among the Hurwitz surfaces. Surfaces 5433B and 8589A also have two classes of short curves which might be systoles, but we have not yet been able to determine if they are.

Table C.1. Length spectrum, genus 3

Primitive translation	Symmetry			n	Length	#
(1 1)	A	D	S	4	3.935946249	21
(2 2)	A	D	S	3	5.208017252	28
(1 1 1 1 1 1 2 1 1 1 1 1 1 2)	A	D	S	1	7.358318437	84
(1 1 2 1 2 2)			S	2	7.609407789	84
(1 1 2 2)			S	3	7.98579225	56
(1 1 1 1 1 2 2 1 1 1 1 1 2 2)	A	D	S	1	8.205600187	84
(1 1 1 2)		D		4	8.524420056	42
(1 2 2 2)	A	D	S	3	8.694448336	28
(1 1 1 1 1 1 1 1 2 2 2 2 2 2 2)			S	1	9.040477887	168
(1 1 1 1 1 2 1 1 2 1 1 1 2 2 1 2)			S	1	9.173078188	168
(1 1 1 1 1 1 1 2 1 1 2 2 2 1 1 2)		D		1	9.45029721	168
(1 1 1 2 2 2 2 1 1 1 2 2 2 2)	A	D	S	1	9.643831436	84
(1 1 1 2 2 1 2 2 2 2 2 1 2 2)		D		1	9.866002403	168
(1 1 2 1 1 2 2 2 2 1 2 2 2 2)		D		1	9.954731231	168
(1 1 1 2 1 2 2 2)			S	2	10.02643445	84
(1 1 1 1 1 2 1 1 1 1 2 2 2 1 1 1 1 2)		D		1	10.34590843	168
(1 1 1 1 2 1 1 1 2 1 2 2 1 1 2 1 1 2)			S	1	10.4160345	168
(1 1 1 2 1 1 1 2 2 1 1 1 2 1 1 1 2 2)	A	D	S	1	10.51527771	84
(1 1 2 2 2 2 2 1 1 2 2 2 2 2)	A	D	S	1	10.51527771	84
(1 1 1 1 2 1 1 1 2 1 1 1 1 2 2 1 2 2)		D		1	10.56968991	168
(1 1 1 2 2 1 1 1 2 2 2 1 1 2 2 2)			S	1	10.56968991	168
(1 1 1 1 2 1 1 1 2 2 1 1 2 2 1 1 1 2)			S	1	10.63250749	168
(1 1 1 1 2 2 2 1 1 2 1 2 2 2 2 2)			S	1	10.87325845	168
(1 1 1 1 1 1 2 1 1 2)			S	2	10.9188537	84
(1 1 1 1 1 1 2 1 1 1 1 2 1 1 1 1 1 1 2 2)			S	1	10.99645644	168
(1 1 1 1 1 2 1 1 1 2 1 1 2 1 1 2 1 1 1 2)		D		1	10.99645644	168
(1 1 1 1 2 1 2 2 1 1 2 2 1 1 1 1 2 2)			S	1	11.04735513	168
(1 1 1 2 2 2 1 2 2 1 2 2 1 2 2 2)		D		1	11.04735513	168
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 2)			S	1	11.28308806	168
(1 1 1 2 1 1 2 1 2 2 2 1 1 2 1 1 2 2)				1	11.28308806	336
(1 1 1 1 1 1 2 1 1 2 1 2 2 2 2 1 2 2)				1	11.32726506	336
(1 1 1 2 1 1 2 2 2 1 1 1 2 1 1 2 2 2)	A			1	11.39075867	168

Table C.2. Length spectrum, genus 7

Primitive translation	Symmetry			n	Length	#
(1 2 2 2)	A	D	S	2	5.796298891	126
(1 1 1 1 1 1 1 2)		D		2	8.303943294	252
(1 1)	A	D	S	9	8.85587906	28
(1 1 2 1 1 2 2 1 1 2 1 1 2 2)	A	D	S	1	8.85587906	252
(1 1 1 1 1 1 2 2)			S	2	9.209465397	252
(1 1 1 1 2 1 1 1 2 1 1 1 1 2 1 1 1 2)	A	D	S	1	9.721826845	252
(1 1 1 1 1 1 2 1 1 2 1 1 1 1 2 2 1 2)			S	1	10.21836713	504
(1 1 1 1 1 2 1 2 2 2 2 2 2 2 2 1 2)		D		1	10.61581822	504
(1 1 1 1 2 2 2 2)			S	2	10.70291713	252
(1 1 1 1 1 2 1 1 2 1 1 1 2 1 1 1 2 1 1 2)		D		1	11.02126698	504
(1 1 1 2 2 1 1 2 2 2 2 2 1 1 2 2)		D		1	11.02126698	504
(1 1 1 1 2 1 1 2 1 2 2 1 1 2 2 2 1 2)	A			1	11.17826694	504
(1 1 1 1 1 2 2 1 1 1 2 1 1 1 2 1 1 1 2 2)		D		1	11.49166195	504
(1 1 1 2 1 1 2 1 2 2 2 1 1 2 1 2 2 2)		D		1	11.49166195	504
(1 1 1 1 2 1 1 1 2 2)				2	11.59259778	504
(1 1 1 1 2 1 1 1 1 2 1 1 1 1 2 2 1 1 2 2)			S	1	11.71172024	504
(1 2 2 1 2 2 2 1 2 2 2 2 1 2 2 2)			S	1	11.71172024	504
(1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 2 1 2 2 1 2)			S	1	11.9560228	504
(1 1 2 2 2 1 2 2 2 2 2 2 1 2 2 2)			S	1	11.9560228	504
(1 1 1 2 2 1 2 2 2 1 1 1 2 2 1 2 2 2)	A			1	12.05583676	504
(1 1 1 1 1 2 1 1 1 1 2 1 1 2 1 1 2 1 1 1 1 2)		D		1	12.13243115	504
(1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2)		D		1	12.13243115	504
(1 1 1 2 2 1 2 2 1 2 2 1 1 1 2 2 2 2)			S	1	12.13243115	504
(1 1 1 1 2 2 2 1 2 2 1 1 2 2 1 2 2 2)			S	1	12.13243115	504
(1 1 1 1 1 2 1 1 2 2 1 1 1 2 1 1 2 2 2 2)				1	12.41565658	1008
(1 1 1 1 2 1 1 2 2 2)				2	12.47983926	504
(1 1 1 1 1 1 1 1 2 1 1 2 2 2 2 2 2 1 1 2)			S	1	12.55709177	504
(1 1 1 2 2 2 1 1 1 2 2 2 2 1 2 2 2 2)		D		1	12.55709177	504
(1 1 1 1 1 2 1 1 2 1 1 2 1 1 2 1 1 1 1 1 2 2)			S	1	12.59053565	504
(1 1 1 1 1 1 2 2 2 1 1 1 2 1 2 2 1 2 2 2)			S	1	12.59053565	504
(1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 2 1 2 2 2 2)				1	12.72050521	1008
(1 1 1 1 1 2 2 1 2 2 2 1 1 2 1 1 1 2 2 2)				1	12.72050521	1008

Table C.3. Length spectrum, genus 14A

Primitive translation	Symmetry			n	Length	#
(1 1)	A	D	S	6	5.903919373	91
(1 2 2 1 2 2)	A	D	S	2	8.403613863	273
(1 1 1 1 2 2 2 1 1 1 1 2 2 2)	A	D	S	1	8.950308066	546
(1 2 2 2 2 2)		D		2	9.308028493	546
(1 1 1 2 1 1 2 2 1 1 2 2 2 2 1 2)			S	1	10.31389083	1092
(1 1 1 1 2 1 1 1 2 1 1 1 1 2 1 1 2 1 1 2)		D		1	11.11869033	1092
(1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 1 1 2)	A	D	S	1	11.27422128	546
(1 1 1 1 2 1 1 1 1 2 2 1 1 1 2 1 1 1 1 2 2)		D		1	11.58771796	1092
(1 1 1 1 1 2 1 1 2 2)				2	11.68983026	1092
(1 1 1 2 2 2)		D		3	11.80783875	364
(2 2)	A	D	S	7	12.15204026	78
(1 1 1 1 1 1 1 1 1 2 2 1 1 1 1 1 1 1 1 1 2 2)	A	D	S	1	12.15204026	546
(1 1 1 2 1 1 1 2 1 1 2 1 1 1 2 1 1 1 2 1 1 2)	A	D	S	1	12.15204026	546
(1 1 1 1 1 2 1 1 1 1 2 1 1 1 1 1 2 1 2 2 1 2)			S	1	12.22952729	1092
(1 1 1 2 1 1 2 1 1 1 2 2 2 2 2 2 2 2)			S	1	12.22952729	1092
(1 1 1 1 1 1 2 1 1 2 1 1 1 2 2 1 1 1 2 1 1 2)			S	1	12.51193467	1092
(1 1 1 1 1 1 2 1 1 1 2 2 2 1 2 2 2 2 1 2)				1	12.51193467	2184
(1 1 1 1 1 1 1 2 1 1 1 2)		D		2	12.57686532	546
(1 1 1 1 2 2 1 1 2 2)			S	2	12.57686532	546
(1 1 1 2 2 1 1 2 1 1 2 2 2 1 1 2 1 1 2 2)		D		1	12.81683683	1092
(1 1 1 1 1 1 2 1 2 2 1 1 1 1 2 2 2 2 2 2)			S	1	12.8174894	1092
(1 1 1 1 1 2 1 2 2 2 1 2 2 1 1 1 2 2 2 2)			S	1	12.93950641	1092
(1 1 1 1 2 1 1 1 1 2 1 2 2 1 1 2 1 1 2 2 1 2)	A	D	S	1	13.03166141	546
(1 1 1 1 2 1 2 2 2 1 1 2 2 2 2 1 1 1 2 2)				1	13.08191796	2184
(1 1 1 1 1 1 1 1 1 2 1 1 2 1 1 1 1 1 1 1 2 2 1 2)			S	1	13.14162148	1092
(1 1 1 2 1 1 2 2 2 2 1 1 2 1 1 1 2 2 2 2)			S	1	13.14162148	1092
(1 1 1 1 1 1 1 1 2 1 1 2 2 1 2 2 1 2 2 1 1 2)			S	1	13.27077669	1092
(1 1 1 1 1 1 2 1 1 1 1 1 2 2 2 1 1 2 2 1 2 2)				1	13.36913886	2184
(1 1 1 1 1 1 2 1 2 2 1 2 2 2 1 1 2 1 1 1 2 2)				1	13.53374266	2184
(1 1 1 1 1 1 2 2 1 1 1 1 1 1 2 2 2 1 1 2 2 2)			S	1	13.57295031	1092
(1 1 1 1 1 2 1 1 1 2 2 2 2 1 2 2 1 1 2 1 1 2)				1	13.57295031	2184
(1 1 1 1 1 1 1 2 2 2 2 1 1 1 1 1 1 1 2 2 2 2)	A	D	S	1	13.61980729	546

Table C.4. Length spectrum, genus 14B

Primitive translation	Symmetry			n	Length	#
(1 1)	A	D	S	7	6.887905935	78
(1 1 2 1 1 2)	A	D	S	2	7.085420866	273
(1 1 1 1 1 2)		D		3	9.464473368	364
(1 1 1 1 2 1 2 2)			S	2	9.520865811	546
(1 2 2 1 2 2 2 1 2 2 1 2 2 2)	A	D	S	1	10.02314704	546
(2 2)	A	D	S	6	10.4160345	91
(1 1 1 2 2 1 2 2)		D		2	10.4160345	546
(1 1 1 1 1 1 2 2 2 1 1 1 1 1 1 2 2 2)	A	D	S	1	10.89656541	546
(1 1 1 1 1 2 1 1 2 1 2 2 2 2 1 1 2 2)				1	11.39614285	2184
(1 1 1 1 2 1 1 1 2 2 2 2 2 2 1 1 1 2)			S	1	11.41779596	1092
(1 1 1 2 1 1 1 2 1 1 2 2 1 1 1 2 2 1 1 2)		D		1	11.81328881	1092
(1 1 1 2 2 2 1 1 2 2 1 2 2 1 1 2 2 2)		D		1	12.20261065	1092
(1 1 1 1 1 1 1 1 2 1 1 1 2 1 1 2 1 1 1 1 2 2)				1	12.21710418	2184
(1 1 2 2 1 1 2 2 2 1 1 2 2 1 1 2 2 2)	A	D	S	1	12.36015633	546
(1 1 1 1 1 2 1 2 2 1 2 2 1 1 1 1 2 2 2 2)				1	12.67452083	2184
(1 1 1 2)		D		6	12.78663008	182
(1 1 1 1 1 1 2 1 1 1 1 2)			S	2	12.78663008	546
(1 2 2 2 2 2 2 2)		D		2	12.78663008	546
(1 1 1 1 1 2 2 1 1 1 2 2 2 2 2 1 1 1 2 2)		D		1	12.89517474	1092
(1 1 1 2 1 1 1 2 1 1 1 2 2 2 2 1 2 2 2 2)		D		1	12.89517474	1092
(1 1 1 1 1 1 1 1 2 2 1 2 2 2 1 1 2 2 2 2)				1	12.90543703	2184
(1 1 1 1 1 1 1 2 1 1 2 1 1 1 1 2 1 1 1 1 2 1 1 2)		D		1	13.14006626	1092
(1 1 1 1 2 2 1 1 2 2 1 2 2 1 2 2 1 1 2 2)			S	1	13.14006626	1092
(1 1 1 1 2 2 1 1 2 1 1 1 2 2 1 2 2 2 2 2)				1	13.14914854	2184
(1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 1 1 1 2)	A	D	S	1	13.24010094	546
(1 1 1 2 1 1 2 1 1 2 2 1 1 1 2 1 1 2 1 1 2 2)	A			1	13.24010094	1092
(1 1 1 1 2 1 1 1 2 2 1 1 1 2 1 1 1 1 2 2 2 2)			S	1	13.32517289	1092
(1 1 1 1 2 2 1 2 2 1 1 2 1 2 2 2 2 1 2 2)			S	1	13.32517289	1092
(1 1 1 1 1 2 1 1 1 1 2 2 1 1 1 1 2 1 1 1 1 1 2 2)			S	1	13.60063104	1092
(1 1 1 1 1 1 1 2 1 1 1 2 2 1 1 2 1 1 1 1 1 1 2 2)				1	13.60063104	2184
(1 1 1 1 2 1 1 2 1 1 1 1 2 1 2 2 2 1 1 2 2 2)				1	13.60063104	2184
(1 1 1 1 1 1 1 2 1 1 2 2)				2	13.6719171	1092

Table C.5. Length spectrum, genus 14C

Primitive translation	Symmetry			n	Length	#
(1 1 1 2)		D		3	6.393315042	364
(1 1 1 1 2 1 1 2)			S	2	8.978514237	546
(1 1 2 2 1 2 2 1 1 2 2 1 2 2)	A	D	S	1	9.499396371	546
(1 1 1 1 1 2 2 2)		D		2	9.877525419	546
(1 1 1 1 1 1 1 1 1 2 1 2 2 2 2 2 1 2)		D		1	10.86841978	1092
(1 1 1 1 2 2)			S	3	10.88194798	364
(1 1 1 1 1 1 1 2 2 1 1 2 2 2 1 1 2 2)		D		1	11.26708217	1092
(1 1 1 2 2 2 2 2)		D		2	11.36503351	546
(1 1 2 1 1 2 2 2 2 2 1 2 2 2 2 2)		D		1	11.68259818	1092
(1 1 1 2 1 2 2 2 2 1 1 1 2 1 2 2 2 2)	A			1	11.83088335	1092
(2 2)	A	D	S	7	12.15204026	78
(1 1 1 1 1 1 2 2 2 1 2 2 2 2 1 2 2 2)			S	1	12.15204026	1092
(1 1 1 1 1 1 1 1 1 1 1 2)		D		2	12.25279933	546
(1 1 2 2 2 2 2 2)			S	2	12.25279933	546
(1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2)		D		1	12.36531286	1092
(1 1 1 1 1 1 1 2 1 2 2 1 2 2 2 1 2 2 1 2)		D		1	12.36531286	1092
(1 1 1 1 1 1 2 1 1 1 1 1 2 2 1 1 2 1 1 1 2 2)				1	12.61565716	2184
(1 1 1 1 2 2 2 2 1 1 2 1 2 2 2 2 2 2)			S	1	12.61565716	1092
(1 1 1 1 1 1 2 1 1 2 2 1 1 1 1 1 1 2 1 1 2 2)	A			1	12.70993456	1092
(1 1 1 1 1 2 1 1 1 2 1 2 2 1 1 1 2 1 2 2 1 2)	A	D	S	1	12.70993456	546
(1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 2 2 1 1 2)			S	1	12.78663008	1092
(1 1 1 1 2 1 1 1 1 2 1 1 1 2 2 1 2 2 1 1 1 2)		D		1	12.79182531	1092
(1 1)	A	D	S	13	12.79182531	42
(1 1 1 1 2 1 1 1 2 1 1 1 2 1 1 2 1 1 1 2 1 1 1 2)			S	1	13.07019765	1092
(1 1 1 2 2 1 1 2 1 2 2 2 1 1 2 2 1 1 2 2)				1	13.07019765	2184
(1 1 1 1 1 1 1 1 1 1 2 2)			S	2	13.13881819	546
(1 1 1 2 1 1 2 2 2 2)				2	13.13881819	1092
(1 1 1 1 1 1 2 1 1 1 2 1 1 1 2 2 1 1 1 2 1 1 1 2)			S	1	13.21178642	1092
(1 1 1 1 1 1 1 1 2 2 1 2 2 2 2 2 2 1 2 2)			S	1	13.21178642	1092
(1 1 1 1 1 2 1 1 2 1 1 1 1 1 2 2 1 2 2 1 2 2)			S	1	13.24939675	1092
(1 1 1 1 1 1 2 2 2 2 1 1 2 2 1 1 2 2 2 2)			S	1	13.37923607	1092
(1 1 1 2 2 1 2 2 2 1 2 2 1 1 2 1 1 2 2 2)				1	13.37923607	2184

Table C.6. Length spectrum, genus 17 (chiral)

Primitive translation	Symmetry			n	Length	#
(1 1 2 1 2 2)*			S	2	7.609407789	336
(1 1)	A	D	S	8	7.871892498	84
(1 1 2 2)			S	3	7.98579225	224
(1 1 1 1 1 2 1 1 2 1 1 1 2 2 1 2)			S	1	9.173078188	672
(2 2)	A	D	S	6	10.4160345	112
(1 1 1 2 2 1 1 1 2 2 2 1 1 2 2 2)			S	1	10.56968991	672
(1 1 1 1 2 2 2 1 1 2 1 2 2 2 2 2)*			S	1	10.87325845	672
(1 1 1 1 1 1 2 1 1 2)			S	2	10.9188537	336
(1 1 1 1 1 1 2 1 1 1 1 2 1 1 1 1 1 1 2 2)*			S	1	10.99645644	672
(1 2 2 1 2 2 2 2)			S	2	11.80783875	336
(1 1 1 1 1 1 1 2 1 1 1 2 1 1 2 2 1 2 2 2)				1	11.8911705	1344
(1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 2 1 1 1 1 1 2)			S	1	11.91870295	672
(1 1 1 1 2 2 1 1 1 2 1 1 1 2 1 2 2 1 2 2)			S	1	12.07908726	672
(1 1 2 1 1 2 1 1 2 2 2 1 2 2 1 2 2 2)			S	1	12.07908726	672
(1 1 1 2 1 1 1 2 2 2)		D		2	12.14744034	672
(1 1 1 1 1 1 1 2 1 1 1 1 2 2 1 1 2 2 2 2)*				1	12.1762261	1344
(1 1 1 1 2 2 2 1 1 1 2 1 1 2 1 1 1 2 2 2)			S	1	12.35268137	672
(1 1 1 1 1 1 2 1 1 1 2 2 2 2 2 2 1 1 1 2)			S	1	12.37867777	672
(1 1 1 1 1 1 2 1 1 1 2 2 2 2 2 2 1 1 1 2)*			S	1	12.37867777	672
(1 1 1 1 1 2 2 1 2 2)		D		2	12.4002864	672
(1 1 2 1 1 2 1 1 2 2)*			S	2	12.4002864	336
(1 1 1 1 1 2 1 1 1 2 2 1 1 1 1 2 1 1 2 1 1 2)				1	12.50329223	1344
(1 1 1 1 1 1 1 1 2 1 2 2 2 1 2 2 2 1 2 2)			S	1	12.50329223	672
(1 1 1 1 1 2 2 1 1 2 2 1 1 1 1 1 2 2 2 2)			S	1	12.58222696	672
(1 2 2 1 2 2)	A	D	S	3	12.60542079	224
(1 1 1 1 2 1 1 2 2 2 2 1 1 2 1 1 2 2 1 2)			S	1	12.67648347	672
(1 1 1 1 2 1 1 1 1 2 1 1 2 2 2 2 2 1 2 2)				1	12.77033577	1344
(1 1 1 1 1 1 1 1 2 2 2 1 1 1 1 1 1 1 1 2 2 2)	A	D	S	1	12.8563103	672
(1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 1 2 2 1 1 1 2)*			S	1	13.0183164	672
(1 1 1 1 2 1 1 2 1 1 2 1 1 1 2 1 2 2 2 1 1 2)			S	1	13.03698743	672
(1 1 1 1 1 1 2 1 2 2 2 2 1 1 1 2 2 2 2 2)				1	13.12742756	1344
(1 1 1 1 1 1 2 1 1 2 2 2 2 1 1 1 1 1 1 2 2)				1	13.1709396	1344

Table C.7. Length spectrum, genus 118

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 1 1 2 1 2 2 1 1 1 2 2 1 2)	A	D	S	1	10.45126176	4914
(1 1 1 1 2 1 1 1 1 2)	A	D	S	2	10.85359441	2457
(1 1 1 1 1 1 1 1 1 2 2 1 1 2 1 1 1 2 1 1 2 2)		D		1	12.65211909	9828
(1 1 2 1 2 2 2 2 2 1 1 2 1 2 2 2 2 2)	A	D	S	1	12.79182531	4914
(1 1)	A	D	S	13	12.79182531	378
(1 1 1 2 2 1 1 2 2 2)				2	13.22121447	9828
(1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 1 1 2)	A	D	S	1	13.67234267	4914
(1 1 1 2 1 1 2 1 2 2 2 1 1 1 2 1 1 2 1 2 2 2)	A	D	S	1	13.67234267	4914
(1 1 1 2 1 1 2 1 1 2 1 1 2 1 1 1 2 2 2 2 2 2)			S	1	14.03304384	9828
(1 1 1 2 1 1 2 1 1 1 2 2)			S	2	14.10603446	4914
(1 1 2 2 1 1 2 2 2 2)			S	2	14.10603446	4914
(1 1 1 1 1 1 1 1 1 2)		D		3	15.42202842	3276
(1 1 1 1 1 1 1 2 1 1 1 1 2 2)				2	15.58039058	9828
(1 1 1 2 1 1 2 2 2 1 2 2)				2	15.58039058	9828
(1 1 1 1 1 2 1 1 1 2)		D		3	15.86670269	3276
(1 1 2 1 1 2 2 2)		D		3	15.86670269	3276
(1 1 1 1 1 1 2 2 1 1 1 1 2 2)			S	2	16.46387863	4914
(1 1 2 1 1 2 2 1 2 2 2 2)				2	16.46387863	9828
(1 1 1 1 1 1 2 1 2 2 1 2 2 2)				2	17.34715569	9828
(1 1 1 1 1 2 1 1 2 2 2 2 1 2)				2	17.34715569	9828
(1 1 1 1 1 1 1 1 1 1 2 1 2 2 1 2)			S	2	17.68516338	4914
(1 1 1 1 1 1 2 2 2 1 1 2 2 2)			S	2	17.93698634	4914
(1 1 1 1 2 1 2 2 1 2 2 1 2 2)				2	17.93698634	9828
(1 1 1 2 1 1 1 2 2 2)		D		3	18.22116051	3276
(1 1 1 1 1 1 1 1 1 2 1 1 1 2 1 1 1 2)		D		2	18.82006541	4914
(1 1 1 1 1 2 1 1 1 1 2 1 1 2 2 2)				2	18.82006541	9828
(1 1 1 1 1 1 1 2 2 2 2 2 2 2)		D		2	18.82006541	4914
(1 1 1 1 2 1 2 2 2 2 2 1 2 2)				2	18.82006541	9828
(1 1 2 1 1 2 2 1 2 2 2 1 2 2)		D		2	18.82006541	4914
(1 1 1 2 2 1 2 2 1 2 2 1 2 2)		D		2	18.82006541	4914
(1 1 1 1 1 2 1 1 2 2 1 1 2 2 1 2)				2	19.23726539	9828
(1 1 1 2 2 2 1 1 1 2 2 2 2 2)		D		2	19.23726539	4914

Table C.8. Length spectrum, genus 146A

Primitive translation	Symmetry			n	Length	#
(2 2)	A	D	S	5	8.680028754	1218
(1 1 1 2 1 1 2 1 1 2)		D		2	11.41115221	6090
(1 1 1 1 1 1 1 2 1 1 2 1 1 1 1 1 1 1 2 1 1 2)	A	D	S	1	11.87644894	6090
(1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 1 1 1 2 2 1 1 2)		D		1	13.65834203	12180
(1 1 1 1 1 1 2 1 2 2 1 2)			S	2	13.77581187	6090
(1 1)	A	D	S	14	13.77581187	435
(1 1 1 1 1 1 2 1 1 1 2 1 1 2 1 2 2 1 1 2 2 1 1 2)				1	14.09197923	24360
(1 1 1 1 2 1 1 2 2 2 2 1 1 1 1 2 1 1 2 2 2 2)	A			1	14.2246	12180
(1 1 1 1 1 2 1 2 2 2 1 2 2 1 2 2 1 2 2 2 1 2)	A	D	S	1	14.2246	6090
(1 1 1 1 1 2 1 2 2 1 2 2)				2	14.66016601	12180
(1 1 2 2 1 2 2 2 2 2)				2	14.66016601	12180
(1 1 1 1 1 1 2 1 1 1 1 1 2 2)				2	15.54399934	12180
(1 1 1 1 1 1 1 1 2 2)			S	3	16.7589434	4060
(1 1 2 2 1 2 2 2)				3	16.7589434	8120
(1 1 1 1 1 1 1 1 2 2 1 2 2 2)				2	17.01743092	12180
(1 1 1 1 1 2 1 1 2 2 1 1 2 2)				2	17.01743092	12180
(1 1 1 2 1 2 2 2 2 2 2 2)				2	17.01743092	12180
(1 1 1 2 1 1 2 1 1 2 2 1 2 2)				2	17.43472112	12180
(1 1 1 1 1 2 1 1 1 1 2 2 1 1 1 2)				2	17.90061613	12180
(1 1 1 2 1 1 2 1 2 2 1 2 2 2)				2	17.90061613	12180
(1 1 1 1 1 1 2 1 1 2 2 2 2 2)				2	17.90061613	12180
(1 1 1 1 2 2 2 1 1 1 1 2 2 2)	A	D	S	2	17.90061613	3045
(1 1 1 1 2 2 1 1 2 2 1 1 2 2)			S	2	17.90061613	6090
(1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 2)	A	D	S	2	18.62359538	3045
(1 1 2 1 1 2 1 2 2 2 2 2 2 1 2)	A	D	S	2	18.62359538	3045
(1 1 1 1 2 1 1 2 2 2)				3	18.71975889	8120
(1 1 1 1 1 1 2 1 1 2 1 1 1 2 2 2)				2	18.78369854	12180
(1 1 1 1 1 2 1 1 2 1 1 2 1 1 2 2)				2	18.78369854	12180
(1 1 1 1 2 1 2 2 2 2 1 2 2 2)				2	18.78369854	12180
(1 1 2 1 2 2)			S	5	19.02351947	2436
(1 1 1 1 1 1 1 2 2 1 1 1 1 2 2 2)				2	19.12165172	12180
(1 1 1 2 1 1 1 2 2 2 2 2 2 2)		D		2	19.12165172	6090

Table C.9. Length spectrum, genus 146B

Primitive translation	Symmetry			n	Length	#
(1 1 1 2)		D		5	10.65552507	2436
(1 1 2 1 2 2 2 2)			S	2	11.06168746	6090
(1 1 1 2 2 1 1 2 2 1 1 1 2 2 1 1 2 2)	A	D	S	1	11.53150714	6090
(1 1 1 1 1 1 1 2)		D		3	12.45591494	4060
(1 1 1 1 1 2 2 1 2 2 1 2 2 2 1 2 2 1 2 2)		D		1	13.31222768	12180
(1 1 1 1 1 1 1 1 2 1 2 2)			S	2	13.42810514	6090
(1 1 1 1 1 2 1 1 2 1 1 2)		D		2	13.42810514	6090
(1 1 1 2 1 1 2 2 1 2 2 1 1 1 2 1 1 2 2 1 2 2)	A			1	13.87828404	12180
(1 1 1 1 2 1 1 2 1 1 2 2)				2	14.31273607	12180
(1 1)	A	D	S	15	14.75979843	406
(1 2 2 2 2 1 2 2 2 2)	A	D	S	2	15.19674706	3045
(1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 2)			S	2	16.67034521	6090
(1 1 1 1 1 2 1 2 2 1 1 1 2 2)			S	2	16.67034521	6090
(1 1 1 2 2 2 1 1 2 2 2 2)				2	16.67034521	12180
(1 1 1 1 2 1 1 1 2 1 1 1 2 1 1 2)				2	17.553585	12180
(1 1 1 1 1 1 1 2 1 2 2 2 2 2)				2	17.553585	12180
(1 1 1 2 1 1 2 2 1 1 1 2 2 2)				2	17.553585	12180
(1 1 2 2 2 1 1 2 2 2 2 2)		D		2	17.553585	6090
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 2)				2	17.553585	12180
(1 1 1 2 1 1 1 2 2 2)		D		3	18.22116051	4060
(1 1 1 1 1 2 1 1 1 2 1 1 1 2 2 2)				2	18.4367025	12180
(1 1 1 1 2 2 1 1 2 2 2 1 2 2)				2	18.4367025	12180
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2)			S	2	18.77466551	6090
(1 1 2 1 1 2 1 2 2 2 2 1 2 2)				2	18.77466551	12180
(1 1 1 2 1 1 2 2 2 2 2 2 1 2)			S	2	18.77466551	6090
(1 1 1 1 1 1 1 1 2 1 1 1 1 2 1 1 1 2)				2	19.02645976	12180
(1 1 1 1 1 1 1 2 1 1 1 1 2 2 2 2)				2	19.02645976	12180
(1 1 1 1 1 1 1 2 2 1 1 2 1 1 2 2)		D		2	19.02645976	6090
(1 1 1 1 2 1 1 1 2 1 1 2 2 2 1 2)				2	19.02645976	12180
(1 1 1 2 2 1 1 2 2 2 1 2 2 2)				2	19.02645976	12180
(1 1 2 2 1 1 2 2 1 2 2 1 2 2)			S	2	19.02645976	6090
(1 1 1 1 2 1 1 1 2 1 1 1 1 2 1 1 1 2)	A	D	S	2	19.44365369	3045

Table C.10. Length spectrum, genus 146C

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 2 2 1 2 2 1 1 1 1 2 2 1 2 2)	A	D	S	1	11.44155775	6090
(1 1 1 1 1 1 1 2 2 2)		D		2	11.85925243	6090
(1 1 1 1 2 1 2 2 1 2)			S	2	11.85925243	6090
(1 1 2 1 1 2 2 2 2 1 1 2 1 1 2 2 2 2)	A	D	S	1	12.3197837	6090
(1 1 1 1 2 1 1 2 1 1 1 2 2 2 2 1 1 1 2 1 1 2)			S	1	13.25086927	12180
(1 1 2 2)			S	5	13.30965375	2436
(1 1 1 1 1 2 1 1 1 1 2 2 2 2 1 2 2 2 1 1 2 2)				1	14.10291541	24360
(1 1 1 1 1 1 1 1 1 1 1 1 1 2)		D		2	14.22206376	6090
(1 2 2 1 2 2 1 2 2 2)		D		2	14.22206376	6090
(1 1)	A	D	S	15	14.75979843	406
(1 1 1 1 1 1 1 1 1 1 1 1 2 2)			S	2	15.10612642	6090
(1 2 2 2 1 2 2 2 2 2)		D		2	15.10612642	6090
(1 1 1 1 1 2)		D		5	15.77412228	2436
(1 1 1 1 1 2 1 1 1 2 2 1 1 2)				2	15.98977262	12180
(1 1 1 1 2 2 2 1 1 2 2 2)			S	2	15.98977262	6090
(1 1 1 1 2 2 2 2)			S	3	16.05437569	4060
(1 1 1 1 1 1 1 1 1 1 2 2 2 2)			S	2	16.57977301	6090
(1 1 1 1 2 1 1 2 1 1 2 2 1 2)			S	2	16.57977301	6090
(1 1 1 1 1 1 1 1 1 1 2 1 1 1 2 2)				2	17.46302868	12180
(1 1 1 1 1 2 1 1 2 1 2 2 2 2)				2	17.46302868	12180
(1 1 1 1 2 1 1 1 2 2 1 2 2 2)				2	17.46302868	12180
(1 1 1 1 1 1 2 1 1 1 2 2 1 1 1 2)			S	2	17.88028855	6090
(1 1 1 1 1 2 1 2 2 2)				3	18.09133088	8120
(1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 2)				2	18.34615638	12180
(1 1 1 1 1 1 1 2 1 1 1 2 2 2 1 2)				2	18.34615638	12180
(1 1 1 1 1 2 1 1 2 1 1 1 2 2 1 2)			S	2	18.34615638	6090
(1 1 1 1 1 1 2 1 2 2 2 2 2 2)				2	18.34615638	12180
(1 1 2 1 1 2 2 1 1 2 2 1 2 2)				2	18.34615638	12180
(1 1 1 1 1 1 1 2 1 1 1 2)		D		3	18.86529797	4060
(1 1 1 1 2 2 1 1 2 2)			S	3	18.86529797	4060
(1 1 1 1 1 1 1 2 1 1 1 1 1 1 2 1 1 2)				2	19.2292018	12180
(1 1 1 1 2 1 1 1 2 1 1 2 2 1 2 2)				2	19.2292018	12180

Table C.11. Length spectrum, genus 411A

Primitive translation	Symmetry			n	Length	#
(1 1)	A	D	S	10	9.839865622	1722
(1 1 1 1 2 1 1 1 1 2 2 1 1 1 1 2 1 1 1 1 2 2)	A	D	S	1	12.67294552	17220
(1 1 1 1 2 2 1 2 2 2)				2	13.10159389	34440
(1 1 1 2 1 1 2 1 1 2 1 1 1 2 1 2 2 2 2 2 1 2)	A	D	S	1	13.55331725	17220
(1 2 2 2)	A	D	S	5	14.49074723	3444
(1 1 1 2 1 1 2 1 2 2 2 2)				2	15.46099963	34440
(1 1 1 1 1 1 1 1 1 2 2 1 2 2)		D		2	16.34452403	17220
(1 1 1 1 1 2 1 1 1 2 2 2 1 2)				2	16.34452403	34440
(1 1 2 1 1 2 1 2 2 2 2 2)				2	16.34452403	34440
(1 1 2 2 2 2)			S	4	17.56583867	8610
(1 1 1 1 1 1 1 2 1 1 1 1 2 2 1 2)				2	17.81766584	34440
(1 1 1 1 2 2 1 1 2 1 1 2 2 2)				2	17.81766584	34440
(1 1 1 1 1 2 2 1 2 2)		D		3	18.6004296	11480
(1 1 2 1 1 2 1 1 2 2)			S	3	18.6004296	11480
(1 1 1 1 2 1 1 1 1 2 1 1 1 2 2 2)				2	18.70075609	34440
(1 1 1 1 1 1 1 2 1 1 1 2 2 1 2 2)				2	18.70075609	34440
(1 1 1 1 2 1 1 2 1 1 2 1 1 1 2 2)				2	18.70075609	34440
(1 1 1 1 1 2 2 1 1 2 2 2 2 2)				2	18.70075609	34440
(1 1 1 1 2 1 2 2 2 2)				3	19.42083127	22960
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2)				2	19.58377744	34440
(1 1 1 1 1 2 1 1 1 2 1 1 1 1 2 1 1 2)				2	19.58377744	34440
(1 1 2 2 1 2 2 1 2 2 1 2 2 2)				2	19.58377744	34440
(1 1 1 2 2 2)		D		5	19.67973124	6888
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 2)		D		2	20.46675449	17220
(1 1 1 1 1 1 2 1 1 1 1 1 1 2 1 2 2 2)				2	20.46675449	34440
(1 1 1 1 1 2 1 1 2 2 1 1 2 2 2 2)				2	20.46675449	34440
(1 1 1 1 2 2 1 1 2 2 1 1 1 2 2 2)				2	20.46675449	34440
(1 1 1 2 1 1 2 1 2 2 1 2 2 1 2 2)				2	20.46675449	34440
(1 1 1 1 2 1 2 2 2 1 1 2 2 2 1 2)			S	2	20.46675449	17220
(1 1 1 1 1 1 1 2 1 1 1 1 2 1 1 2 2 2)				2	20.80467817	34440
(1 1 1 1 1 1 2 2 1 1 1 1 2 1 1 1 2 2)				2	20.80467817	34440
(1 1 1 1 1 2 2 1 2 2 1 2 2 1 2 2)		D		2	20.80467817	17220

Table C.12. Length spectrum, genus 411B

Primitive translation	Symmetry			n	Length	#
(1 1 2 2)			S	4	10.647723	8610
(1 1 2 1 1 2 2 1 2 2)		D		2	13.01666223	17220
(1 1 1 1 2 2 1 1 1 2 2 1 1 1 1 2 2 1 1 1 2 2)	A	D	S	1	13.46883019	17220
(1 1 1 1 1 2 1 1 2 2 1 2)				2	13.90168943	34440
(1 1 1 1 1 1 2 1 1 2 1 1 2 2)				2	16.2597911	34440
(1 1 1 2 2 1 2 2 1 2 2 2)				2	16.2597911	34440
(1 2 2 2 2 2 2 2 2 2 2)		D		2	16.2597911	17220
(1 1 1 1 1 1 2 1 1 2)			S	3	16.37828055	11480
(1 1 1 1 1 2 2 1 1 1 1 2 2 2)				2	17.14310896	34440
(1 1 2 2 1 2 2 1 2 2 2 2)				2	17.14310896	34440
(1 1 1 1 2 2)			S	5	18.13657996	6888
(1 2 2 2 2 2)		D		4	18.61605699	8610
(1 1 1 1 1 1 1 1 1 2 1 2 2 1 2 2)				2	18.61605699	34440
(1 1 1 1 2 1 1 1 2 1 1 2 1 1 2 2)				2	18.61605699	34440
(1 1 1 1 1 2 1 1 2 2 2 2 2 2)				2	18.61605699	34440
(1 1 2 1 1 2 1 2 2 2)			S	3	19.28834936	11480
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 2 2)				2	19.4990837	34440
(1 1 1 1 1 1 1 2 1 1 2 2 1 2 2 2)				2	19.4990837	34440
(1 1 1 1 2 1 1 1 2 2 1 1 1 2 2 2)				2	19.4990837	34440
(1 1 1 1 2 2 1 1 2 2 2 2 2 2)				2	19.4990837	34440
(1 1 1 1 1 1 1 1 2 2 1 1 1 1 1 1 2 2)			S	2	20.38206421	17220
(1 1 1 1 1 2 1 1 2 1 1 1 2 1 1 1 2 2)				2	20.38206421	34440
(1 1 1 1 1 1 2 1 1 1 1 2 1 1 2 2 1 2)				2	20.38206421	34440
(1 1 1 1 2 2 1 1 2 1 1 1 2 2 2 2)				2	20.38206421	34440
(1 1 1 2 2 1 1 2 1 1 2 2 1 1 2 2)				2	20.38206421	34440
(1 1 1 1 2 1 1 1 2 1 1 1 2 1 2 2 1 2)			S	2	20.38206421	17220
(1 1 1 1 1 1 2 1 2 2 1 2)			S	3	20.66371781	11480
(1 1)	A	D	S	21	20.66371781	820
(1 1 1 1 1 1 1 2 1 1 1 2 2 1 1 1 2 2)				2	20.71998885	34440
(1 1 1 2 1 2 2 2 1 1 2 2 1 1 2 2)				2	20.71998885	34440
(1 1 1 2 1 1 2 2 2 1 2 2 2 1 1 2)		D		2	20.71998885	17220
(1 1 1 1 1 1 1 2)		D		5	20.75985823	6888

Table C.13. Length spectrum, genus 411C

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 1 1 1 1 2 1 2 2 1 1 1 1 1 2 2 1 2)	A	D	S	1	12.43189638	17220
(1 1 1 1 2 2 2 2 2 1 1 1 1 2 2 2 2 2)	A	D	S	1	12.43189638	17220
(1 1 2 1 1 2 1 2 2 2)			S	2	12.85889957	17220
(1 1 1 1 1 1 1 2 2)			S	3	13.8141981	11480
(1 1 2 1 2 2)			S	4	15.21881558	8610
(1 1 1 1 1 1 2 1 1 1 2 1 1 2)				2	15.21881558	34440
(1 1 1 1 1 2 2 1 1 2 2 2)				2	15.21881558	34440
(1 1 1 1 1 2 1 1 2 1 1 1 2 2)				2	16.10242073	34440
(1 1 1 1 1 2 1 1 2 2 2 1 2 2)				2	17.5756383	34440
(1 1 2 2 1 1 2 2 2 2 2 2)			S	2	17.5756383	17220
(1 1 1 1 2 2)			S	5	18.13657996	6888
(1 1 1 1 1 1 1 1 1 1 2 2 1 1 2 2)			S	2	18.45875337	17220
(1 1 1 1 2 1 1 1 1 2 1 1 2 2 1 2)			S	2	18.45875337	17220
(1 1 1 1 1 2 1 1 1 2 2 2 1 1 1 2)		D		2	18.45875337	17220
(1 1 1 2 1 1 2 2 1 1 2 2 2 2)				2	18.45875337	34440
(1 1 1 1 2 2 1 2 2 1 1 2 2 2)				2	18.45875337	34440
(1 1 1 2 2 2 1 1 2 1 1 2 2 2)		D		2	18.45875337	17220
(1 1 1 2 1 1 1 2 2 1 1 2 1 1 2 2)		D		2	19.34179069	17220
(1 1 1 1 1 1 1 2 1 1 2 2 2 2 1 2)				2	19.34179069	34440
(1 1 1 1 1 2 1 1 2 2 1 2 2 1 1 2)		D		2	19.34179069	17220
(1 1 1 2 1 2 2 2 2 2 2 1 2 2 2)				2	19.34179069	34440
(1 1 2 1 1 2 2 1 2 2)		D		3	19.52499335	11480
(1 1 1 2 2 2)		D		5	19.67973124	6888
(1 1)	A	D	S	20	19.67973124	861
(1 1 1 1 1 2 1 1 1 1 1 2 2 2 2 2)		D		2	19.67973124	17220
(1 1 1 1 2 1 1 2 1 1 1 1 2 2 2 2)			S	2	19.67973124	17220
(1 1 1 1 1 1 1 1 1 2 1 1 1 2 1 1 2 2)				2	19.93151107	34440
(1 1 1 1 1 2 1 2 2 2 1 1 1 2 2 2)				2	19.93151107	34440
(1 1 2 1 1 2 2 1 2 2 2 2 2 2 2)				2	19.93151107	34440
(1 2 2 2)	A	D	S	7	20.28704612	2460
(1 1 1 1 1 1 1 1 1 2)		D		4	20.56270456	8610
(1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 1 2 2)				2	20.81447539	34440

Table C.14. Length spectrum, genus 474A

Primitive translation	Symmetry			n	Length	#
(1 1)	A	D	S	11	10.82385218	1806
(1 1 1 2 1 1 2 2 1 1 2 1 1 1 2 1 1 2 2 1 1 2)	A	D	S	1	13.16694649	19866
(1 1 1 1 1 2 1 1 1 1 1 2 2)				2	13.59840295	39732
(1 1 1 1 1 2 2 1 2 2 2 1 1 1 1 1 2 2 1 2 2 2)	A			1	14.0478706	39732
(1 1 1 2 1 2 2 2)			S	3	15.03965168	13244
(1 1 1 1 2 1 1 1 2 2 1 1 1 2)			S	2	15.95693873	19866
(1 1 1 1 2 1 1 2 2 2 2 2)				2	15.95693873	39732
(1 1 1 1 1 1 1 2 1 1 1 2 1 1 1 2)		D		2	16.84032537	19866
(1 1 1 1 1 1 1 1 2 1 2 2 2 2)				2	16.84032537	39732
(1 1 1 1 2 1 1 2 1 1 1 2 2 2)				2	16.84032537	39732
(1 1 1 1 1 2 2 2 2 2 2 2 2)		D		2	16.84032537	19866
(1 1 1 1 1 1 2 1 2 2)			S	3	17.20188207	13244
(1 1 2 1 1 2 1 1 2 2 1 1 2 2)			S	2	17.72353726	19866
(1 2 2 1 2 2 1 2 2 2 2 2)		D		2	17.72353726	19866
(1 1 1 1 1 1 1 1 1 1 1 2 2 1 2 2)		D		2	18.31333794	19866
(1 1 1 2 2 2 2 2 2 2 2 2)		D		2	18.31333794	19866
(2 2)	A	D	S	11	19.09606326	1806
(1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 2)				2	19.19638582	39732
(1 1 1 1 2 1 1 1 2 2 1 1 2 2 1 2)				2	19.19638582	39732
(1 1 2 2 2 2 2 2 2 2 2 2)			S	2	19.19638582	19866
(1 1 2 1 2 2 2 1 2 2 2 1 2 2)			S	2	19.19638582	19866
(1 1 1 1 1 1 1 1 2 1 1 2)			S	3	19.31482312	13244
(1 1 1 1 1 2 1 1 2 2 2 1 1 1 2 2)				2	19.61357522	39732
(1 1 1 1 1 1 1 2 1 1 2 1 1 1 1 1 2 2)				2	20.07937993	39732
(1 1 1 1 1 1 1 1 2 1 1 1 1 2 2 1 1 2)				2	20.07937993	39732
(1 1 1 1 1 1 2 1 1 1 1 1 2 1 2 2 1 2)				2	20.07937993	39732
(1 1 1 1 1 2 1 1 2 2 1 2 2 1 2 2)				2	20.07937993	39732
(1 1 1 1 2 1 1 2 2 2 1 1 2 2 1 2)				2	20.07937993	39732
(1 1 1 2 1 1 2 2 2 2 2 2 2 2)				2	20.07937993	39732
(1 1 1 2 1 1 1 2 1 2 2 2 1 2 2 2)			S	2	20.07937993	19866
(1 1 1 1 2 1 2 2 2 1 1 1 2 2 2 2)				2	20.80225475	39732
(1 1 1 1 1 1 1 2 1 1 1 2 2 1 2 2 1 2)				2	20.96233946	39732

Table C.15. Length spectrum, genus 474B

Primitive translation	Symmetry			n	Length	#
(1 1 2 1 1 2)	A	D	S	3	10.6281313	6622
(1 1 2 1 2 2 2 1 2 2)			S	2	13.40037453	19866
(1 1 1 1 1 1 1 1 1 2 1 1 2 1 1 1 1 1 1 1 1 1 2 1 1 2)	A	D	S	1	13.85067507	19866
(1 1 1 1 1 2 1 2 2 2 2 1 1 1 1 1 2 1 2 2 2 2)	A			1	13.85067507	39732
(1 1 1 1 2 1 1 2 2 1 1 2)			S	2	14.28502968	19866
(1 1 1 2 1 1 1 2 1 2 2 2)			S	2	14.28502968	19866
(1 1 1 2 1 1 2 1 1 1 2 1 1 2)	A	D	S	2	15.75923143	9933
(1 1 1 1 1 1 1 2 1 2 2 1 2 2)				2	16.64266893	39732
(1 1 1 1 1 1 2 1 1 2 2 2 1 2)				2	16.64266893	39732
(1 1 1 1 1 1 1 2 1 1 1 1 1 1 2 2)				2	17.5259135	39732
(1 1 1 2 1 1 2 1 1 1 2 2 2 2)			S	2	17.5259135	19866
(1 1 2 1 1 2 2 2 2 2 2 2)		D		2	17.5259135	19866
(1 1 1 1 1 1 2 1 1 1 1 1 1 2 1 1 1 2)		D		2	18.99879274	19866
(1 1 1 1 1 1 1 2 1 1 2 2 1 1 2 2)				2	18.99879274	39732
(1 1 1 1 2 1 1 2 1 1 1 2 1 2 2 2)			S	2	18.99879274	19866
(1 1 1 2 1 1 2 2 1 2 2 2 2 2)				2	18.99879274	39732
(1 1 2 2 1 2 2 1 1 2 2 1 2 2)	A	D	S	2	18.99879274	9933
(1 1 1 2 1 1 2 1 1 1 2 2 1 1 2 2)			S	2	19.41598744	19866
(1 1 2 1 2 2 2 2 1 2 2 1 2 2)			S	2	19.41598744	19866
(1 1 1 1 1 1 2 1 1 1 1 2 1 1 2 1 1 2)				2	19.88179691	39732
(1 1 1 1 1 1 2 1 1 1 1 1 2 2 1 1 1 2)				2	19.88179691	39732
(1 1 1 1 1 1 2 1 2 2 1 2 2 1 2 2)				2	19.88179691	39732
(1 1 1 1 2 1 2 2 2 2 2 2 2 2)				2	19.88179691	39732
(1 1 2 1 1 2 2 2 1 2 2 2 2 2)				2	19.88179691	39732
(1 1 1 1 1 1 2 1 1 1 2 2)				3	20.30768236	26488
(1 1 1 2 1 2 2 2 2 2)				3	20.30768236	26488
(1 1 1 1 1 1 1 1 2 1 1 1 1 2 2 1 2 2)				2	20.76476291	39732
(1 1 1 1 1 1 1 2 1 1 2 1 1 2 1 1 2 2)				2	20.76476291	39732
(1 1 1 1 2 2 1 1 2 1 2 2 1 2 2 2)				2	20.76476291	39732
(1 1 1 1 2 1 2 2 1 2 2 1 1 2 2 2)				2	20.76476291	39732
(1 1 1 1 1 1 2 1 2 2 2 2 2 1 2 2)				2	20.76476291	39732
(1 1 1 2 2 1 2 2 2 2 2 2 2 2)				2	20.76476291	39732

Table C.16. Length spectrum, genus 474C

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 2 1 1 1 1 2 1 1 1 1 1 2 1 1 1 1 2)	A	D	S	1	11.74745067	19866
(1 2 2 2 2 2)		D		3	13.96204274	13244
(1 1 1 2 2 1 1 1 2 2 2 1 1 1 2 2 1 1 1 2 2 2)	A	D	S	1	14.09511537	19866
(1 1 1 1 1 1 2 2 1 1 2 2)			S	2	14.53028733	19866
(1 1 2 1 2 2 2 2 2 2)			S	2	14.53028733	19866
(1 1 1 1 1 1 1 2 1 1 2 1 1 2)		D		2	15.41418349	19866
(1 1 1 1 2 1 2 2 1 2 2 2)				2	15.41418349	39732
(1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 2)		D		2	16.88767398	19866
(1 1 1 1 1 1 2 1 1 2 2 1 2 2)				2	16.88767398	39732
(1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2)		D		2	17.7708785	19866
(1 1 1 1 1 1 1 1 2 1 1 1 2 2 1 2)				2	17.7708785	39732
(1 1 1 1 1 2 1 1 2 1 1 1 2 1 1 2)		D		2	17.7708785	19866
(1 1 1 1 2 1 1 2 1 1 2 2 2 2)				2	17.7708785	39732
(1 1 1 1 1 1 1 2 1 2 2 1 1 1 2 2)				2	18.65397332	39732
(1 1 1 1 2 1 1 1 2 2 1 1 2 1 1 2)				2	18.65397332	39732
(1 1 2 1 1 2 1 2 2 2 1 2 2 2)			S	2	18.65397332	19866
(1 2 2 2 2 1 2 2 2 2 2 2)			S	2	18.65397332	19866
(2 2)	A	D	S	11	19.09606326	1806
(1 1 1 1 1 2 1 1 1 1 2 2 2 2 1 2)				2	19.24372015	39732
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2)		D		2	20.126712	19866
(1 1 1 1 1 1 2 1 1 2 1 1 1 1 1 1 2 2)			S	2	20.126712	19866
(1 1 1 1 2 1 1 1 2 1 1 1 2 1 1 1 2 2)				2	20.126712	39732
(1 1 1 1 2 1 1 2 1 2 2 1 1 2 2 2)				2	20.126712	39732
(1 1 1 1 1 2 1 1 1 2 2 2 2 1 2 2)				2	20.126712	39732
(1 1 1 1 1 1 2 1 1 2 2 2 2 2 1 2)				2	20.126712	39732
(1 1 1 1 1 2 2 1 1 2 2 1 1 2 2 2)				2	20.54388238	39732
(1 1 1 2 1 1 1 2 2 1 2 2 1 2 2 2)				2	20.54388238	39732
(1 1 1 1 1 1 2 1 1 1 1 2 1 1 1 2 2 2)				2	20.6405463	39732
(1 1 1 1 1 1 2 1 2 2 1 2)			S	3	20.66371781	13244
(1 1)	A	D	S	21	20.66371781	946
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2)			S	2	21.00967007	19866
(1 1 1 1 1 1 2 1 1 1 2 1 1 2 1 2 2 2)				2	21.00967007	39732

Table C.17. Length spectrum, genus 2091A (chiral)

Primitive translation	Symmetry			n	Length	#
(1 1)	A	D	S	11	10.82385218	7980
(1 1 2 2)			S	5	13.30965375	17556
(1 1 1 2 1 1 1 2 1 2 2 2)			S	2	14.28502968	43890
(1 1 2 1 2 2 2 2 2 2)*			S	2	14.53028733	43890
(1 1 1 1 2 1 2 2 1 2 2 2 1 2)				2	17.64213591	87780
(1 1 1 2 1 2 2 1 2 2 1 1 2 2)*			S	2	17.95702847	43890
(1 1 1 1 2 2)			S	5	18.13657996	17556
(1 1 1 1 1 1 1 1 1 1 2 2 1 2 2)		D		2	18.31333794	87780
(1 1 2 1 1 2 2 1 1 2 2 1 2 2)*				2	18.34615638	87780
(1 2 2 2 2 1 2 2 2 2 2 2)*			S	2	18.65397332	43890
(1 1 1 1 2 1 1 2 1 1 1 2 1 2 2 2)			S	2	18.99879274	43890
(1 1 2 1 2 2)			S	5	19.02351947	17556
(1 1 1 1 1 2 1 1 1 1 2 1 2 2 2 2)				2	19.30197093	87780
(1 1 1 1 2 1 2 2 2 2)				3	19.42083127	58520
(1 1 1 1 2 1 1 1 2 2 1 1 1 2 2 2)				2	19.4990837	87780
(1 1 1 1 1 2 1 2 2 1 1 2 2 2 1 2)				2	19.6209006	87780
(1 1 1 1 1 1 2 1 1 2 2 2 2 2 1 2)*				2	20.126712	87780
(1 2 2 2)	A	D	S	7	20.28704612	12540
(1 1 1 1 1 1 1 2 1 1 2 1 1 1 2 2 1 2)*				2	20.33767583	87780
(1 1 2 2 2 1 2 2 2 2 1 2 2 2)*			S	2	20.45606623	43890
(1 1 1 2 1 1 2 2 2 1 1 2 2 1 2 2)				2	20.92925464	87780
(1 1 1 1 2 2 1 1 1 2 2 2 1 2 2 2)				2	20.95056301	87780
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 2 2 2)				2	21.00034004	87780
(1 1 1 2 2 1 1 2 2 1 1 1 2 2 2 2)*			S	2	21.17574254	43890
(1 1 1 1 1 2 1 1 1 2 2 2)				3	21.2562626	58520
(1 1 1 2)		D		10	21.31105014	17556
(1 1 1 1 1 1 2 1 1 1 2 1 1 1 1 2 1 1 1 2)*			S	2	21.36299087	43890
(1 1 1 1 2 1 1 2 1 1 2 2 1 1 2 1 1 2)*			S	2	21.37833421	43890
(1 1 1 2 1 1 2 2 1 2 2 2 2 1 2 2)*				2	21.47729711	87780
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2)				2	21.55202406	87780
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2)*				2	21.55202406	87780
(1 1 1 1 1 1 1 2 2 1 2 2)		D		3	21.56144906	58520

Table C.18. Length spectrum, genus 2091B (chiral)

Primitive translation	Symmetry			n	Length	#
(1 1 1 2)		D		6	12.78663008	29260
(1 1 1 1 2 1 1 1 2 1 1 2 1 1 1 1 2 1 1 2 1 1 1 2)			S	1	13.28868756	87780
(1 1 2 2)			S	5	13.30965375	17556
(1 1)	A	D	S	15	14.75979843	5852
(1 1 1 1 1 1 1 1 1 1 2 1 2 2)*			S	2	15.39317069	43890
(1 1 1 2 1 2 2 2 1 2 2 2)*			S	2	15.82116967	43890
(1 1 2 2)*			S	6	15.9715845	14630
(1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2)			S	2	16.80722773	43890
(1 1 1 1 1 2 1 1 2 2 1 2 2 2)				2	17.51353668	87780
(1 1 1 2 1 1 2 1 1 1 2 2 2 2)*			S	2	17.5259135	43890
(1 1 1 1 2 2 1 1 1 1 2 2 2 2)*			S	2	17.95702847	43890
(1 1 2 1 1 2 2 1 1 2 2 1 2 2)*				2	18.34615638	87780
(1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 2)*			S	2	18.63297575	43890
(1 1 1 1 2 1 1 1 1 2 1 1 1 2 2 2)				2	18.70075609	87780
(1 1 1 1 2 1 1 2 2 2)*				3	18.71975889	58520
(1 1 1 1 1 2)		D		6	18.92894674	29260
(1 1 1 2 1 2 2 1 2 2)*			S	3	18.97950394	29260
(1 1 1 1 1 2 2 1 2 2 2 2 2 2)*				2	19.28412118	87780
(1 1 2 1 1 2 1 1 2 1 1 2 1 1 2 2)*			S	2	19.48546489	43890
(1 1 1 1 1 1 2 1 1 1 2 1 1 2 1 1 1 2)*			S	2	19.5235075	43890
(1 1 1 1 1 2 2 1 1 2 1 1 1 2 2 2)*				2	19.553756	87780
(1 1 1 1 2 1 1 2 1 1 1 1 2 2 2 2)			S	2	19.67973124	43890
(1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2)			S	2	20.04629407	43890
(1 1 1 1 1 1 2 1 1 2 1 1 1 1 2 2 1 2)*			S	2	20.43673425	43890
(1 1 1 1 1 1 2 1 2 2 1 2)*			S	3	20.66371781	29260
(1 1 1 1 2 1 1 1 2 2 2 2 2 2 1 2)				2	20.7523892	87780
(1 1 1 1 1 1 2 2 2 1 2 2 1 2 2 2)			S	2	20.92397439	43890
(1 1 1 1 1 1 2 1 1 1 2 1 1 2 2 2 1 2)				2	20.94109391	87780
(1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 1 1 1 2)*				2	20.9886751	87780
(1 1 1 2 1 1 2 1 1 1 2 1 1 2 2 1 1 2)*			S	2	21.04722572	43890
(1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 1 2)				2	21.05644715	87780
(1 1 1 1 1 1 1 1 1 2 1 2 2 1 1 2 2 2)				2	21.40583425	87780

Table C.19. Length spectrum, genus 2091C (chiral)

Primitive translation	Symmetry			n	Length	#
(1 1)	A	D	S	10	9.839865622	8778
(1 1 1 1 2 1 2 2)*			S	3	14.28129872	29260
(1 1 1 1 2 1 1 1 2 2 1 1 1 2)*			S	2	15.95693873	43890
(1 1 2 2)			S	6	15.9715845	14630
(1 1 1 1 1 1 1 1 1 2 2 1 2 2)		D		2	16.34452403	87780
(1 1 1 1 2 2 2 2 2 2 2 2)*			S	2	17.64213591	43890
(1 1 1 1 1 1 2 1 1 1 2 2 1 1 1 2)			S	2	17.88028855	43890
(1 1 1 1 1 2 1 1 2 1 1 1 1 1 2 2)			S	2	18.10029091	43890
(1 1 1 1 2 1 1 1 1 2 1 1 2 2 1 2)			S	2	18.45875337	43890
(1 1 1 1 1 2 2 1 2 2)		D		3	18.6004296	58520
(1 1 1 1 2 1 2 2 2 2 2 2 1 2)			S	2	18.67073277	43890
(1 1 2 1 1 2 2 1 2 2 1 2 2 2)				2	18.92023623	87780
(1 1 1 1 1 1 1 1 2 1 2 2 2 2 1 2)*			S	2	19.07583592	43890
(1 1 1 1 1 1 1 1 2 1 1 2)			S	3	19.31482312	29260
(1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2)				2	19.58377744	87780
(1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2)*				2	19.58377744	87780
(1 1 1 1 1 1 2 2 2 2 2 2 2 2)			S	2	19.59115755	43890
(1 1 1 1 2 1 2 2 1 2 2 1 1 1 2 2)*				2	19.95883604	87780
(1 1 1 1 1 2 2 1 1 1 2 2 1 2 2 2)				2	20.26766009	87780
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 2)		D		2	20.46675449	87780
(1 1 1 1 2 1 2 2 1 1 2 2 1 2 2 2)				2	20.69181686	87780
(1 1 1 1 1 2 2 1 2 2 1 2 2 1 2 2)		D		2	20.80467817	87780
(1 1 1 1 1 2 1 2 2 2 2 1 1 2 2 2)				2	20.87561255	87780
(1 1 1 2 1 1 2 2 2 1 1 2 2 1 2 2)*				2	20.92925464	87780
(1 1 1 1 1 1 2 1 1 2 1 1 1 2 1 2 2 2)			S	2	20.94109391	43890
(1 2 2 1 2 2)	A	D	S	5	21.00903466	17556
(1 1 1 1 2 1 1 2 1 1 1 2 1 1 1 2 2 2)*				2	21.11402109	87780
(1 1 1 2 1 2 2 2 1 1 2 1 2 2 2 2)				2	21.13937982	87780
(1 1 1 1 2 2 1 1 1 2 1 1 1 2 1 1 2 2)				2	21.22062782	87780
(1 1 1 1 1 2 1 1 1 2 1 1 1 1 2 2 2 2)*				2	21.31968359	87780
(1 1 1 1 1 2 1 1 1 1 1 2 2 1 1 2 2 2)*				2	21.53369744	87780
(1 1 1 1 2 1 1 2 1 1 2 2 1 1 1 1 2 2)*				2	21.56666044	87780

Table C.20. Length spectrum, genus 2091D

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 2)		D		5	15.77412228	35112
(1 1 1 1 1 1 1 1 1 1 1 1 2 2 2)		D		2	15.8021961	87780
(1 1 1 1 2 1 1 1 1 2)	A	D	S	3	16.28039162	29260
(1 1 1 2 1 1 2 1 1 2 2 2 1 2)			S	2	17.09870695	87780
(1 1 2 1 1 2 1 1 2 2 1 1 2 2)			S	2	17.72353726	87780
(1 1 1 1 1 2 1 1 2 1 1 1 2 1 1 2)		D		2	17.7708785	87780
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2)		D		2	18.15867163	87780
(1 1)	A	D	S	19	18.69574468	4620
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2)			S	2	19.04173162	87780
(1 1 1 1 2 2 1 2 2 2 2 2 1 2 2)			S	2	19.04173162	87780
(1 2 2 2 2 2 2 2 2)		D		3	19.17994513	58520
(1 1 1 2 1 1 2 2 1 1 1 2 2 1 1 2)		D		2	19.35938546	87780
(1 1 1 2 1 1 2 2 1 1 2 1 1 1 2 2)			S	2	19.39864046	87780
(1 1 1 2 2 2)		D		5	19.67973124	35112
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2)			S	2	20.51443761	87780
(1 1 1 1 2 1 2 2 1 1 2 2 1 2 2 2)				2	20.69181686	175560
(1 1 1 1 2 1 1 1 2 2 1 1 2 2 1 1 1 2)			S	2	21.26501499	87780
(1 1 1 2 1 2 2 1 2 2 2 1 1 2 2 2)			S	2	21.33042329	87780
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 2 2)				2	21.39738496	175560
(1 1 1 1 1 2 2 1 1 1 2 2)		D		3	21.63448908	58520
(1 1 1 2 2 1 2 2 2 1 1 1 2 2 2 2)				2	21.69741576	175560
(1 1 1 1 2 2)			S	6	21.76389596	29260
(1 1 1 1 1 1 2 1 1 1 1 2 2 2 2 2 1 2)				2	22.02231373	175560
(1 1 1 1 2 1 1 2 2 2 1 1 1 2 1 1 2 2)				2	22.15183563	175560
(1 1 1 1 1 2 1 1 1 2 1 1 1 1 2 2 1 1 1 2)				2	22.17451392	175560
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 2)				2	22.28031443	175560
(1 1 1 1 1 2 2 1 2 2 1 1 1 1 1 2 2 2)		D		2	22.28031443	87780
(1 1 2 1 1 2 2 1 2 2 2 2 2 1 2 2)		D		2	22.32651496	87780
(1 1 1 1 2 1 1 2 1 1 2 2 1 2 2 2 1 2)			S	2	22.36850752	87780
(1 1 1 1 1 2 1 1 1 1 2 2 2 2 1 1 2 2)				2	22.37513697	175560
(1 1 1 1 1 2 1 2 2 1 2 2 2 1 1 1 2 2)				2	22.52267715	175560
(1 1 1 2 1 1 1 2 1 1 2 1 1 2 2 2 2 2)				2	22.6932324	175560

Table C.21. Length spectrum, genus 2131A

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 2)		D		4	12.61929782	44730
(1 1 1 1 2 1 1 1 2 1 1 1 2 2)				2	15.86347718	178920
(1 1 2 1 1 2 1 1 2 2 2 2)			S	2	15.86347718	89460
(1 1 1 2 2 1 1 1 2 2 2 2)			S	2	15.86347718	89460
(1 1 1 1 1 1 1 2 1 1 2 1 1 2 2)				2	18.21992178	178920
(1 1 1 2 1 1 1 2 1 1 1 2 1 1 2 2)				2	18.21992178	178920
(1 1 1 1 2 1 2 2 2 1 1 2 2 2)				2	18.21992178	178920
(1 1 2 1 2 2)			S	5	19.02351947	35784
(1 1 1 1 1 2 1 1 1 2 1 1 1 2 1 1 1 2)		D		2	19.10297686	89460
(1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 1 2)				2	19.10297686	178920
(1 1 1 1 1 1 1 1 2 1 1 2 2 2 2)				2	19.10297686	178920
(1 1 1 2 2 1 2 2 2 1 1 2 2 2)				2	19.10297686	178920
(1 1 1 2)		D		9	19.17994513	19880
(1 1 1 1 1 1 2 1 1 1 1 2)			S	3	19.17994513	59640
(1 2 2 2 2 2 2 2)		D		3	19.17994513	59640
(1 1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 2 2)				2	19.9859756	178920
(1 1 1 1 1 1 1 1 2 1 1 1 2 1 2 2 1 2)			S	2	19.9859756	89460
(1 1 2 2 1 1 2 2 1 2 2 2 2 2)				2	19.9859756	178920
(1 1 1 1 1 2 2 2 2 2)		D		3	20.04828424	59640
(1 1 1 2 1 1 1 2 1 2 2 1 2 2 2 2)			S	2	20.32390535	89460
(1 1 2 1 2 2 2 1 2 2 2 2 2 2)			S	2	20.32390535	89460
(1 1 1 1 2 1 1 1 2 2 1 1 1 1 2 1 1 2)				2	20.57567824	178920
(1 1 1 1 1 2 1 1 1 2 1 1 1 2 1 2 2 2)				2	20.57567824	178920
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 2 2)				2	21.45862408	178920
(1 1 1 1 1 2 1 1 1 1 1 2 1 1 2 2 2 2)				2	21.45862408	178920
(1 1 1 1 2 1 1 2 1 1 1 1 2 1 1 2 2 2)				2	21.45862408	178920
(1 1 1 1 1 2 2 1 1 1 1 2 1 1 1 2 2 2)				2	21.45862408	178920
(1 1 1 1 1 2 1 1 2 1 1 2 2 1 1 1 2 2)				2	21.45862408	178920
(1 1 1 1 2 1 1 1 2 1 1 1 2 1 2 2 2 2)				2	21.45862408	178920
(1 1 1 1 2 2 1 2 2 1 2 2 2 1 2 2)				2	21.45862408	178920
(1 1 1 1 1 1 2 1 2 2 2 2)				3	22.32606554	119280
(1 1 1 1 1 1 2 2 1 2 2 1 1 1 1 2 2 2)				2	22.34155258	178920

Table C.22. Length spectrum, genus 2131B

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 1 1 1 2 1 1 2)			S	2	12.87654874	89460
(1 1 1 1 2 1 1 1 2 1 2 2 1 2)			S	2	16.12002446	89460
(1 1 1 1 1 1 2 2 2 2 2 2)			S	2	16.12002446	89460
(1 1 1 1 1 1 2 2 2 1 2 2 2 2)				2	18.47634938	178920
(1 1 1 2 1 1 2 2 2 2 1 1 2 2)				2	18.47634938	178920
(1 1 1 2 1 2 2 1 2 2 1 2 2 2)				2	18.47634938	178920
(1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 1 1 2)		D		2	19.35938546	89460
(1 1 1 1 1 1 2 1 1 1 2 2 2 1 2 2)				2	19.35938546	178920
(1 1 1 1 2 1 1 2 1 1 1 2 2 1 2 2)				2	19.35938546	178920
(1 1 1 1 1 2 1 1 2 1 2 2 1 1 2 2)				2	19.35938546	178920
(1 1 1 2 1 1 2 2 1 1 1 2 2 1 1 2)		D		2	19.35938546	89460
(1 1 2 1 1 2 2 2 1 1 2 2 2 2)				2	19.35938546	178920
(1 1 1 2 1 1 2 2 1 1 2 1 1 2 2 2)				2	20.24237199	178920
(1 1 1 1 1 1 2 1 1 2 1 1 1 2 1 1 2 2)				2	20.58029832	178920
(1 1 1 1 1 1 2 1 2 2 2 2 2 2 1 2)			S	2	20.58029832	89460
(1 1 1 1 1 1 1 2 1 1 2 1 1 1 1 2 2 2)				2	20.83206901	178920
(1 1 1 1 1 1 2 1 1 1 1 2 2 1 1 1 2 2)				2	20.83206901	178920
(1 1 1 1 2 1 1 1 2 1 2 2 1 1 2 1 1 2)			S	2	20.83206901	89460
(1 1 1 2 1 1 2 2 1 1 2 2 1 2 2 2)				2	20.83206901	178920
(2 2)	A	D	S	12	20.83206901	7455
(1 1 1 2 2 1 2 2)		D		4	20.83206901	44730
(1 1 1 1 1 1 1 1 2 1 1 2 2 2 1 1 2 2)				2	21.715009	178920
(1 1 1 1 1 1 1 2 2 1 1 1 2 2 1 1 2 2)				2	21.715009	178920
(1 1 1 2 1 1 1 2 1 1 2 1 1 2 2 1 2 2)				2	21.715009	178920
(1 1 1 1 1 2 1 1 1 1 2 2 2 1 2 2 1 2)				2	21.715009	178920
(1 1 2 1 1 2 2 1 1 2 2 2 1 2 2 2)				2	21.715009	178920
(1 1 1 1 1 2 2 1 2 2 2 2 2 1 2 2)		D		2	21.715009	89460
(1 1 1 1 1 1 1 1 2 1 1 1 1 2 1 1 1 2 2 2)				2	22.59793374	178920
(1 1 1 1 1 1 2 1 1 2 1 1 1 1 1 2 1 1 2 2)				2	22.59793374	178920
(1 1 1 1 1 1 1 1 1 1 1 2 1 2 2 1 1 1 2 2)				2	22.59793374	178920
(1 1 1 1 1 1 1 2 1 1 1 1 2 1 1 2 2 1 1 2)				2	22.59793374	178920
(1 1 1 1 1 1 1 2 1 1 1 2 1 1 2 1 2 2 1 2)				2	22.59793374	178920

Table C.23. Length spectrum, genus 2131C

Primitive translation	Symmetry			n	Length	#
(1 1 1 2 2 1 1 1 2 2)	A	D	S	2	12.43588706	44730
(1 1 1 2 1 1 1 2 1 1 2 1 1 2)		D		2	15.68062627	89460
(1 1 1 1 1 2 2 2 1 2 2 2)		D		2	15.68062627	89460
(1 1 1 1 1 1 2 1 1 1 2 1 2 2 1 2)			S	2	18.03716605	89460
(1 1 1 1 1 2 1 2 2 2 2 1 2 2)				2	18.03716605	178920
(1 1 1 1 1 1 1 1 1 1 1 2)		D		3	18.37919899	59640
(1 1 2 2 2 2 2 2)			S	3	18.37919899	59640
(1 1 1 1 1 2 1 1 1 1 2 2 1 1 2 2)				2	18.92023623	178920
(1 1 1 2 1 1 2 1 2 2 2 2 2 2)				2	18.92023623	178920
(1 1 1 1 1 1 2 1 1 2 2 1 1 1 2 2)				2	18.92023623	178920
(1 1 1 1 1 2 1 1 1 2 1 2 2 1 2 2)				2	18.92023623	178920
(1 1 2 1 1 2 2 1 2 2 1 2 2 2)				2	18.92023623	178920
(1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 1 1 2)			S	2	20.14117715	89460
(1 1 1 1 1 1 1 1 1 1 2 2 1 1 1 1 2 2)			S	2	20.39295179	89460
(1 1 1 1 1 2 1 1 1 2 1 1 2 1 1 1 2 2)				2	20.39295179	178920
(1 1 1 1 1 1 2 1 1 1 1 2 1 2 2 1 1 2)			S	2	20.39295179	89460
(1 1 1 1 2 1 1 2 1 1 1 2 1 1 2 1 1 2)				2	20.39295179	178920
(1 1 1 1 1 1 2 2 1 1 2 2 2 1 2 2)				2	20.39295179	178920
(1 1 2 1 2 2 2 2 1 2 2 2 2 2)				2	20.39295179	178920
(1 1 1 1 1 2 1 1 2 1 1 1 2 1 1 2 2 2)				2	21.27590228	178920
(1 1 1 1 2 1 1 1 2 2 1 1 2 1 1 1 2 2)				2	21.27590228	178920
(1 1 1 1 2 1 1 2 2 1 1 2 2 2 2 2)				2	21.27590228	178920
(1 1 1 1 2 2 2 1 1 2 1 1 2 2 2 2)				2	21.27590228	178920
(1 1 1 2 1 1 1 2 1 2 2 2 2 2 2 2)				2	21.27590228	178920
(1 1 1 2 1 1 2 1 2 2 2 2 1 2 2 2)				2	21.27590228	178920
(1 1 2 2 2 2)			S	5	21.95729834	35784
(1 1 1 1 1 1 1 1 1 2 1 1 1 2 1 1 2 2 1 2)				2	22.15883377	178920
(1 1 1 1 1 1 1 1 2 1 1 2 1 2 2 2 2 2)				2	22.15883377	178920
(1 1 1 1 1 2 1 1 1 2 2 2 2 1 1 1 2 2)				2	22.15883377	178920
(1 1 1 2 1 1 1 2 1 2 2 2 1 1 1 2 2 2)				2	22.15883377	178920
(1 1 1 2 1 1 2 2 2 1 1 2)		D		3	22.34201974	59640
(1 1 1 1 1 1 2 1 1 1 2 2 1 1 1 1 1 1 2 2)				2	22.74850546	178920

Table C.24. Length spectrum, genus 3404A

Primitive translation	Symmetry			n	Length	#
(1 1 2 2 2 1 1 2 2 2)	A	D	S	2	14.07685081	71463
(1 1 1 1 1 1 1 1 1 2 2 2 2 2)		D		2	17.31801279	142926
(1 1 1 1 2 1 1 2 1 2 2 1 2 2)				2	17.31801279	285852
(1 1 1 2 1 1 2 2 1 2 2 1 1 2)		D		2	17.31801279	142926
(1 1 1 1 2 1 1 2 2 2 2 2 1 2)				2	18.20115782	285852
(1 1 1 1 2 1 1 1 2 1 2 2 1 2 2 2)				2	19.67394363	285852
(1 1 1 1 1 2 1 1 2 1 2 2 2 2 1 2)				2	19.67394363	285852
(1 1 1 2 2 2 1 2 2 1 2 2 2 2)				2	19.67394363	285852
(1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 2)		D		2	20.55691716	142926
(1 1 1 1 1 1 2 1 1 1 2 1 1 2 1 1 2 2)				2	20.55691716	285852
(1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 2 1 2)				2	20.55691716	285852
(1 1 1 1 1 2 2 1 1 2 2 2 1 1 2 2)		D		2	20.55691716	142926
(1 1 1 1 2 1 2 2 1 1 2 1 2 2 2 2)				2	20.55691716	285852
(1 1 1 1 2 1 1 2 1 2 2 2 1 2 2 2)			S	2	20.55691716	142926
(1 1 2 2 2 1 2 2 1 2 2 2 2 2)				2	20.55691716	285852
(1 1 2 2 1 2 2 2 2 2 2 1 2 2)			S	2	20.55691716	142926
(1 1 1 2 1 1 1 2 1 1 2 2)				3	20.93468824	190568
(1 1 1 1 1 1 1 1 2 1 1 1 2 1 1 1 2 1 1 2)				2	21.43986347	285852
(1 1 1 1 1 1 1 2 1 1 1 1 2 2 2 1 2 2)				2	21.43986347	285852
(1 1 1 1 1 2 1 1 1 2 2 2 1 1 1 1 2 2)				2	21.43986347	285852
(1 1 1 2 2 1 1 2 1 2 2 2 2 1 2 2)				2	21.43986347	285852
(1 1 1 1 1 1 1 1 2 1 1 1 2 1 1 1 1 1 2 2)				2	21.77777853	285852
(1 1 1 1 2 1 1 2 2 2 2 2 1 2 2 2)				2	21.77777853	285852
(1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 1 2 2 1 2)				2	22.02954198	285852
(1 1 1 1 2 1 1 1 1 2 1 1 2 2 1 2 2 2)				2	22.02954198	285852
(1 1 1 1 2 1 1 2 1 1 2 1 1 2 2 1 2 2)				2	22.02954198	285852
(1 1 1 1 1 2 1 1 1 2 2 1 2 2 1 1 2 2)				2	22.02954198	285852
(1 1 1 2 2 1 1 2 2 1 1 2 2 2 2 2)				2	22.02954198	285852
(1 1 1 1 1 1 2 2 1 2 2 2)				3	22.58240196	190568
(1 1 1 2 2 1 1 2 1 1 2 2)		D		3	22.58240196	95284
(1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 2 1 1 1 2)				2	22.91246271	285852
(1 1 1 1 1 1 1 1 1 2 2 1 1 1 1 2 1 1 2 2)				2	22.91246271	285852

Table C.25. Length spectrum, genus 3404B

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 1 1 1 2 1 1 1 1 2)			S	2	14.74151436	142926
(1 1 1 1 1 1 2 2 1 1 1 2 2 2)				2	17.09870695	285852
(1 1 1 2 1 1 2 1 1 2 2 2 1 2)			S	2	17.09870695	142926
(1 1 2 2 1 2 2 2 2 1 2 2)			S	2	17.09870695	142926
(1 1 1 1 1 1 1 2 1 1 1 2 2 1 1 2)				2	17.9818807	285852
(1 1 1 1 1 1 2 2 1 1 2 2 2 2)				2	17.9818807	285852
(1 1 1 1 1 1 1 1 1 1 1 2 2 1 1 1 2 2)		D		2	20.33767583	142926
(1 1 1 1 1 1 1 2 1 1 2 1 1 1 2 2 1 2)				2	20.33767583	285852
(1 1 1 1 2 2 2 1 1 1 2 1 1 2 2 2)				2	20.33767583	285852
(1 1 1 1 1 1 2 1 1 2 2 2 2 1 2 2)				2	20.33767583	285852
(1 1 1 2 1 1 2 2 1 1 2 2 1 1 2 2)				2	20.33767583	285852
(1 1 1 1 2 1 1 1 1 2 1 1 1 2 2 1 2 2)				2	21.22062782	285852
(1 1 1 1 2 2 1 1 1 2 1 1 1 2 1 1 2 2)				2	21.22062782	285852
(1 1 1 1 1 1 2 1 1 1 1 1 2 2 2 2 1 2)				2	21.22062782	285852
(1 1 2 1 1 2 2 1 1 2 2 1 1 2 2 2)				2	21.22062782	285852
(1 1 1 1 2 1 1 2 2 2 2 2 2 1 1 2)			S	2	21.22062782	142926
(1 1 1 1 1 1 1 1 1 1 2 1 1 2 1 1 1 1 2 2)				2	22.10356027	285852
(1 1 1 1 1 2 1 1 2 2 1 1 1 2 2 1 2 2)				2	22.10356027	285852
(1 1 1 1 1 1 2 1 2 2 1 1 2 2 1 1 2 2)				2	22.10356027	285852
(1 1 1 1 2 1 1 1 2 2 2 2 1 1 2 1 1 2)				2	22.10356027	285852
(1 1 1 1 1 2 1 2 2 2 1 1 1 2 2 2 1 2)		D		2	22.10356027	142926
(1 1 2 1 1 2 1 2 2 2 2 2 2 2 2 1 2)		D		2	22.10356027	142926
(1 1 1 1 1 1 1 1 1 1 1 2 1 1 2)			S	3	22.26103062	95284
(1 1 1 1 1 2 1 1 1 2 1 1 1 2)		D		3	22.26103062	95284
(1 1 1 1 2 1 2 2 1 1 2 2)			S	3	22.26103062	95284
(1 1 1 2 2 2 2 2 2 2)		D		3	22.26103062	95284
(1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 2 2 1 2)				2	22.44147145	285852
(1 1 1 1 1 1 1 2 2 2 1 1 2 1 1 2 2 2)		D		2	22.44147145	142926
(1 1 1 1 1 2 1 1 1 2 1 1 1 2 1 1 2 2 1 2)				2	22.44147145	285852
(1 1 1 2 2 1 1 2 1 2 2 2 2 2 2 2)				2	22.44147145	285852
(1 1 1 1 1 1 1 1 1 2 1 1 1 2 2 1 1 1 2 2)				2	22.6932324	285852
(1 1 1 2 1 1 1 2 1 1 2 1 1 2 2 2 2 2)				2	22.6932324	285852

Table C.26. Length spectrum, genus 3404C

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 1 1 2 2 2 2 2)		D		2	15.34554839	142926
(1 1 1 1 2 1 1 2 1 1 1 1 2 2)			S	2	16.22911007	142926
(1 1 1 2 2 1 2 2 2 1 2 2)		D		2	16.22911007	142926
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 2)		D		2	18.58538856	142926
(1 1 1 1 2 1 1 1 2 1 1 2 2 1 1 2)				2	18.58538856	285852
(1 1 1 1 2 2 2 1 1 1 2 2 2 2)				2	18.58538856	285852
(1 2 2 2 1 2 2 2 2 2 2 2)		D		2	18.58538856	142926
(1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 2)				2	19.46841728	285852
(1 1 1 1 2 2 2 1 1 2 2 2 2 2)				2	19.46841728	285852
(1 1 1 1 2 1 1 1 2 1 2 2 2 2 1 2)			S	2	19.46841728	142926
(1 1 1 2 2 2 1 2 2 2 1 2 2 2)		D		2	19.46841728	142926
(1 1 1 1 1 1 1 1 1 1 2 2)			S	3	19.70822729	95284
(1 1 1 2 1 1 2 2 2 2)				3	19.70822729	190568
(1 1 1 1 1 1 1 2 1 1 1 1 1 2 2 2 2)				2	20.94109391	285852
(1 1 1 1 1 1 1 2 2 1 1 1 1 2 1 1 2 2)				2	20.94109391	285852
(1 1 1 1 1 1 2 1 1 2 1 1 1 2 1 2 2 2)			S	2	20.94109391	142926
(1 1 1 1 1 1 2 1 1 1 2 1 1 2 2 2 1 2)				2	20.94109391	285852
(1 1 1 2 2 1 1 2 1 1 2 2 2 1 2 2)				2	20.94109391	285852
(1 1 1 1 2 2 1 2 2 1 1 2 2 1 2 2)			S	2	20.94109391	142926
(1 1 1 1 2 2 1 1 2 2 2 2 1 1 2 2)			S	2	21.35825356	142926
(1 1 1 2 1 1 1 2 1 1 2 1 1 2 2 2 1 2)			S	2	21.35825356	142926
(1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 2 2 2)				2	21.82403164	285852
(1 1 1 1 1 1 1 2 2 1 1 1 1 1 2 2 2 2)				2	21.82403164	285852
(1 1 1 1 1 1 1 1 1 2 1 2 2 2 1 2 2 2)				2	21.82403164	285852
(1 1 1 2 1 1 1 2 1 1 2 2 1 1 1 2 2 2)				2	21.82403164	285852
(1 1 1 1 1 1 1 2 1 1 2 2 2 1 2 2 1 2)				2	21.82403164	285852
(1 1 2 2 1 1 2 2 1 1 2 2 1 2 2 2)				2	21.82403164	285852
(1 1 1 1 1 1 1 1 1 1 2 1 1 2 1 1 1 2 2 2)				2	22.70695491	285852
(1 1 1 1 1 1 1 2 1 1 1 2 2 1 1 1 1 1 2 2)				2	22.70695491	285852
(1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 2 2 1 1 2)				2	22.70695491	285852
(1 1 1 1 1 2 1 1 1 1 2 2 2 2 2 2 2 1 2)				2	22.70695491	285852
(1 1 1 1 2 1 1 1 2 2 2 2 1 2 2 1 1 2)				2	22.70695491	285852

Table C.27. Length spectrum, genus 5433A

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 1 2 1 2 2 2 2 2)				2	14.88404369	456288
(1 1 1 2 1 1 2 1 1 2)		D		3	17.11672832	152096
(1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 2)				2	17.24111645	456288
(1 1 1 2 2 2 2 1 2 2 2 2)		D		2	17.24111645	228144
(1 1 1 1 1 1 2 1 1 1 1 2 2 1 1 2)				2	18.12427119	456288
(1 1 1 2 1 1 1 2 2 1 2 2 2 2)				2	18.12427119	456288
(1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 2 2 2)				2	20.48004264	456288
(1 1 1 1 2 1 1 1 2 1 1 1 1 2 1 1 2 2)				2	20.48004264	456288
(1 1 1 1 1 1 1 1 2 1 1 1 1 2 2 2 1 2)				2	20.48004264	456288
(1 1 1 1 1 2 1 1 2 2 2 2 1 1 2 2)				2	20.48004264	456288
(1 1 1 1 2 2 1 1 1 2 2 2 1 1 2 2)				2	20.48004264	456288
(1 1 1 1 1 2 1 2 2 1 2 2 2 1 2 2)				2	20.48004264	456288
(1 1 2 2 1 2 2 2 2 2 1 2 2 2)				2	20.48004264	456288
(1 1 1 1 1 1 2 1 1 1 2 1 1 1 1 2 1 1 1 2)			S	2	21.36299087	228144
(1 1 1 1 1 1 1 2 1 2 2 1 1 1 1 2 2 2)				2	21.36299087	456288
(1 1 1 2 1 1 2 1 2 2 2 2 2 2 1 2 2)				2	21.36299087	456288
(1 1 1 2 1 2 2 2 2 1 1 2 2 1 2 2)				2	21.36299087	456288
(1 1 1 1 1 1 1 2 2 1 1 1 2 2 1 2 2 2)				2	22.24592089	456288
(1 1 1 2 1 1 2 2 2 2 2 2 2 2 1 2)			S	2	22.24592089	228144
(1 1 1 1 2 1 2 2 1 2 2 1 1 2 1 1 2 2)				2	22.5838314	456288
(1 1 1 2 2 1 2 2 2 1 1 2 2 2 2 2)				2	22.5838314	456288
(1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 2)				2	22.83559192	456288
(1 1 1 1 1 1 1 2 1 1 1 1 2 2 1 1 1 1 2 2)				2	22.83559192	456288
(1 1 1 1 1 1 2 1 1 1 2 1 1 1 1 2 1 2 2 2)				2	22.83559192	456288
(1 1 1 2 1 1 2 1 1 1 2 1 1 2 1 1 2 1 1 2)			S	2	22.83559192	228144
(1 1 1 1 2 1 1 2 2 1 1 1 2 2 1 2 2 2)				2	22.83559192	456288
(1 1 1 1 1 2 2 1 1 2 1 2 2 1 1 2 2 2)				2	22.83559192	456288
(1 1 1 1 2 1 1 1 2 2 2 2 2 2 1 1 1 2)			S	2	22.83559192	228144
(1 1 2 1 1 2 2 2 2 1 1 2 2 2 2 2)				2	22.83559192	456288
(1 1 2 1 2 2 2 2 2 1 2 2 1 2 2 2)			S	2	22.83559192	228144
(1 1 1 1 1 2 1 1 1 2 2 1 1 1 1 2 2 1 1 2)				2	23.2527385	456288
(1 1 1 2 1 1 2 1 2 2 1 2 2 2 1 1 2 2)				2	23.2527385	456288

Table C.28. Length spectrum, genus 5433B

Primitive translation	Symmetry			n	Length	#
(1 1 2 2)			S	6	15.9715845	76048
(1 1 1 1 1 2 1 1 1 2 1 1 2 2)				2	15.9715845	456288
(1 1 1 1 2 1 1 1 1 2 2 1 2 2)		D		2	16.85496757	228144
(1 1 2 1 2 2 2 1 2 2 2 2)			S	2	16.85496757	228144
(1 1 1 1 1 2 1 1 2 1 1 2 1 2 2 2)				2	19.21102357	456288
(1 1 1 2 1 1 1 2 1 1 2 1 1 2 2 2)				2	19.21102357	456288
(1 1 1 1 1 1 1 1 2 1 2 2 2 1 2 2)			S	2	19.21102357	228144
(1 1 1 1 1 1 2 1 1 1 2 2 2 2 1 2)				2	19.21102357	456288
(1 1 1 1 1 1 1 1 2 1 1 1 1 2 1 1 2 2)				2	20.09401697	456288
(1 1 1 1 1 1 2 1 1 1 2 1 1 1 2 2 1 2)				2	20.09401697	456288
(1 1 1 1 2 1 1 2 1 2 2 2 1 1 2 2)				2	20.09401697	456288
(1 1 1 2 1 1 2 2 1 2 2 1 1 1 2 2)				2	20.09401697	456288
(1 1 1 1 1 2 1 1 2 2 1 1 1 1 1 2 2 2)				2	21.56666044	456288
(1 1 1 1 1 1 2 1 1 2 2 1 1 2 1 1 2 2)				2	21.56666044	456288
(1 1 1 1 2 1 1 2 1 1 2 2 1 1 1 1 2 2)				2	21.56666044	456288
(1 1 1 1 2 1 1 1 2 1 1 1 2 2 2 2 1 2)				2	21.56666044	456288
(1 1 1 2 2 1 1 2 2 1 2 2 1 2 2 2)				2	21.56666044	456288
(1 1 2 2 2 2 2 1 2 2 2 2 2 2)				2	21.56666044	456288
(1 1 1 1 1 1 2 1 2 2 2 1 1 2 1 1 2 2)				2	21.98381436	456288
(1 1 1 1 1 1 1 2 1 1 1 2 1 1 1 1 1 2 2 2)				2	22.44958729	456288
(1 1 1 1 2 1 1 1 1 2 1 1 1 1 2 1 1 1 2 2)				2	22.44958729	456288
(1 1 1 1 1 1 2 1 1 1 1 1 2 2 2 2 2 2)				2	22.44958729	456288
(1 1 1 1 1 1 1 2 1 1 2 2 1 1 2 2 2 2)				2	22.44958729	456288
(1 1 1 1 2 1 2 2 1 1 2 1 2 2 1 1 2 2)			S	2	22.44958729	228144
(1 1 1 1 1 2 1 2 2 2 1 2 2 1 1 1 2 2)			S	2	22.44958729	228144
(1 1 1 1 2 1 1 1 2 1 2 2 2 2 2 1 1 2)				2	22.44958729	456288
(1 1 1 2 1 1 2 2 2 2 1 2 2 2 2 2)				2	22.44958729	456288
(1 1 1 1 2 1 1 1 1 2 1 1 1 1 2 1 1 2 2 2)				2	23.33250358	456288
(1 1 1 1 1 1 1 2 1 1 2 1 1 1 1 2 2 1 2 2)				2	23.33250358	456288
(1 1 1 1 1 1 1 2 2 1 1 2 1 1 1 2 1 1 2 2)		D		2	23.33250358	228144
(1 1 1 1 1 1 2 1 1 2 1 1 1 2 2 1 1 1 2 2)				2	23.33250358	456288
(1 1 1 1 1 1 1 2 1 1 1 1 2 2 2 2 1 1 1 2)				2	23.33250358	456288

Table C.29. Length spectrum, genus 5433C

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 1 2 2 2 2 2 2 1 1 1 1 1 1 2 2 2 2 2 2)	A	D	S	1	14.38041394	228144
(1 1 1 1 1 1 1 1 2 1 1 1 2 1 1 2)				2	17.17354996	456288
(1 1 1 1 1 2 1 1 1 1 2 1 1 1 1 2)		D		2	17.17354996	228144
(1 1 1 2 2 2 1 2 2 2 2 2)				2	17.17354996	456288
(1 1 1 2 1 2 2 2 2 1 1 1 2 2)				2	18.05671355	456288
(1 1 1 1 1 1 1 1 1 2 1 2 2 2 2 2)				2	19.52951678	456288
(1 1 1 1 1 2 1 1 2 1 1 1 2 2 2 2)				2	19.52951678	456288
(1 1 1 1 1 2 1 1 1 2 2 1 1 1 2 1 1 2)				2	20.41249602	456288
(1 1 1 1 2 1 1 2 2 1 1 1 2 2 2 2)				2	20.41249602	456288
(1 1 1 1 1 1 2 2 1 2 2 1 1 2 2 2)				2	20.41249602	456288
(1 1 2 2 2 1 2 2 2 1 2 2 2 2)				2	20.41249602	456288
(1 1 1 1 1 1 1 2 1 2 2 2)				3	21.07993836	304192
(1 1 2 1 1 2 2 2 2 2)		D		3	21.07993836	152096
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 2)			S	2	21.295446	228144
(1 1 1 1 1 2 1 1 2 1 1 2 1 1 1 2 2 2)				2	21.295446	456288
(1 1 1 1 2 1 1 1 2 2 1 1 1 2 1 1 2 2)				2	21.295446	456288
(1 1 1 1 1 1 2 1 1 1 2 2 2 2 1 1 1 2)			S	2	21.295446	228144
(1 1 1 1 2 2 1 1 2 1 1 2 2 2 2 2)				2	21.295446	456288
(1 1 1 2 1 1 2 2 1 2 2 2 1 2 2 2)				2	21.295446	456288
(1 1 2 2)			S	8	21.295446	57036
(1 1 1 1 1 1 2 1 1 1 1 1 2 2 1 1 1 1 1 2)			S	2	21.8851262	228144
(1 1 1 1 2 1 1 1 2 1 2 2 2 1 2 2 1 2)			S	2	21.8851262	228144
(1 1 1 1 1 1 2 1 1 1 1 1 1 2)	A	D	S	3	22.07495531	76048
(1 1 1 2 1 1 1 2 2 1 2 2)		D		3	22.07495531	152096
(1 1 1 1 2 1 1 2 2 2 1 2)				3	22.07495531	304192
(1 1 1 1 1 1 1 1 2 2 1 1 1 1 2 1 1 1 2 2)				2	22.7680487	456288
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 2 2 1 1 2)				2	22.7680487	456288
(1 1 1 2 1 1 1 2 1 1 2 1 1 2 1 1 2 1 1 2)		D		2	22.7680487	228144
(1 1 1 1 2 1 1 1 2 1 1 1 2 2 2 2 2 2)				2	22.7680487	456288
(1 1 1 1 1 1 2 2 1 1 1 2 1 2 2 2 2 2)				2	22.7680487	456288
(1 1 1 1 1 2 1 2 2 1 1 2 2 1 1 2 2 2)				2	22.7680487	456288
(1 1 1 1 1 1 2 1 2 2 2 2 1 1 1 2 2 2)				2	22.7680487	456288

Table C.30. Length spectrum, genus 7201A (chiral)

Primitive translation	Symmetry			n	Length	#
(1 1)	A	D	S	10	9.839865622	30240
(1 1 1 2 2 1 1 1 2 2 2 2)			S	2	15.86347718	151200
(1 1 1 1 1 1 1 1 1 2 2 1 2 2)		D		2	16.34452403	302400
(1 1 2 1 2 2 2 2 1 1 2 2)			S	2	16.38166385	151200
(1 1 1 1 1 2 1 2 2 1 1 1 2 2)			S	2	16.67034521	151200
(2 2)	A	D	S	10	17.36005751	30240
(1 1 1 1 2 2)			S	5	18.13657996	60480
(1 1 1 1 1 2 2 1 2 2)		D		3	18.6004296	201600
(1 1 2 2)*			S	7	18.63351525	43200
(1 1 1 1 1 2 1 1 2 1 1 2 2 1 1 2)*				2	18.73788153	302400
(1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2)				2	19.58377744	302400
(1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2)*				2	19.58377744	302400
(1 1 1 1 2 1 1 2 2 1 1 2 1 1 2 2)*				2	19.6209006	302400
(1 1 1 1 2 1 1 1 2 1 1 2)*				3	19.92912471	201600
(1 1 1 1 1 1 1 1 2 1 1 1 2 1 2 2 1 2)*			S	2	19.9859756	151200
(1 1 2 1 2 2 2 1 2 2)*			S	3	20.1005618	100800
(1 1 1 1 2 1 2 2 1 2 2 1 2 2 1 2)*			S	2	20.25642557	151200
(1 1 1 1 1 1 2 1 1 2 1 1 1 1 2 2 1 2)			S	2	20.43673425	151200
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 2)		D		2	20.46675449	302400
(1 1 1 2 2 2 1 2 2 2)		D		3	20.50787566	201600
(1 1 1 1 1 2 2 1 2 2 1 2 2 1 2 2)		D		2	20.80467817	302400
(1 1 1 1 2 1 1 1 2 2 2 1 2 2 2 2)				2	20.84179945	302400
(1 2 2 1 2 2)	A	D	S	5	21.00903466	60480
(1 1 1 1 1 2 1 1 1 2 2 2 2 2 2 2)*				2	21.13937982	302400
(1 1 1 2 1 1 2 2 2 2 1 2 2 1 2 2)				2	21.53369744	302400
(1 1 1 1 2 1 1 2 1 1 2 1 2 2 2 1 1 2)*			S	2	21.83350054	151200
(1 1 1 1 1 1 2 2 1 1 1 2 1 2 2 1 2 2)*			S	2	21.85469557	151200
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 2 2)				2	21.93938261	302400
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 2 2 2)*				2	21.93938261	302400
(1 1 2 2 2 2)*			S	5	21.95729834	60480
(1 1 1 1 1 1 1 1 2 1 2 2 2 2 1 1 2 2)*				2	22.16754319	302400
(1 1 1 1 2 1 1 1 1 2 2 2 1 1 1 2 2 2)		D		2	22.2497158	302400

Table C.31. Length spectrum, genus 7201B (chiral)

Primitive translation	Symmetry			n	Length	#
(1 1 2 1 2 2)			S	3	11.41411168	100800
(1 1)	A	D	S	15	14.75979843	20160
(1 1 1 1 2 1 2 2 2 1 2 2)			S	2	15.30694395	151200
(1 1 1 2 1 1 2 2 2 2 1 2)			S	2	15.30694395	151200
(2 2)	A	D	S	10	17.36005751	30240
(1 1 1 1 1 1 2 1 1 1 1 2 1 2 2 2)*				2	18.54681321	302400
(1 1 1 2 1 2 2 2 2 1 2 2 2 2 2)*				2	19.42984449	302400
(1 1 1 1 1 1 1 1 1 1 1 2 2)*			S	3	19.70822729	100800
(1 1 1 2 1 1 1 2 1 1 2 2 1 2 2 2)				2	19.76778335	302400
(1 1 1 2 2 1 1 2 2 2)*				3	19.8318217	201600
(1 1 1 1 2 2 1 1 1 2 1 2 2 1 2 2)*			S	2	19.89717376	151200
(1 1 1 1 2 1 1 2 1 1 2 2 2 2 1 2)				2	20.0195621	302400
(1 1 1 1 1 1 1 2 1 2 2 2 2 1 2 2)*				2	20.0195621	302400
(1 1 1 1 1 1 1 1 1 2 1 1 1 2 1 2 2 2)				2	20.28162525	302400
(1 1 1 2 1 1 2 2 1 1 2 2 2 2 1 2)*			S	2	20.62778166	151200
(1 1 1 1 2 1 1 1 2 1 1 1 1 2 1 2 2 2)*				2	20.90252352	302400
(1 1 1 1 2 1 2 2 2 2 1 2 2 1 2 2)				2	21.31968359	302400
(1 1 1 1 2 1 1 2 1 1 2 2 1 1 2 1 1 2)*			S	2	21.37833421	151200
(1 1 1 1 1 1 1 2 1 1 1 2 1 1 2 2 2 2)				2	21.41634541	302400
(1 1 1 1 1 1 1 2 1 1 1 1 2 2 1 2 2 2)				2	21.41634541	302400
(1 1 1 2 1 1 1 2 1 2 2 2)			S	3	21.42754452	100800
(1 1 1 1 1 1 2 2 1 1 2 2)			S	3	21.795431	100800
(1 1 1 1 2 1 1 2 2 1 1 2 1 1 2 2 1 2)*			S	2	21.88845695	151200
(1 1 2 2 2 2)*			S	5	21.95729834	60480
(1 1 2 2 1 2 2 2 2 2)				3	21.99024901	201600
(1 1 1 1 1 1 1 2 1 1 2 1 1 1 1 1 2 2 1 2)*			S	2	22.32949784	151200
(1 1 1 1 2 1 1 1 2 1 1 2 2 2 2 2 1 2)				2	22.50830648	302400
(1 1 1 1 1 1 1 1 2 1 2 2 2 2 1 2 2 2)				2	22.66838582	302400
(1 1 1 1 2 1 1 2 2 1 1 1 2 2 2 2 1 2)				2	22.66838582	302400
(1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 2 1 2 2)				2	22.67453679	302400
(1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 2 2 2)				2	22.97051898	302400
(1 1 1 1 2 1 1 2 1 2 2 1 2 2 2 2 1 2)				2	23.00629458	302400

Table C.32. Length spectrum, genus 7201C

Primitive translation	Symmetry			n	Length	#
(1 1)	A	D	S	12	11.80783875	25200
(1 1 1 2)		D		6	12.78663008	100800
(1 1 2 1 1 2 2 2)		D		3	15.86670269	201600
(1 1 1 1 1 2 2 1 1 1 1 1 2 2)	A	D	S	2	16.41120037	151200
(1 1 1 1 2 1 1 1 2 1 1 2 1 1 1 2)			S	2	17.602969	302400
(1 1 1 1 1 1 2 2 2 1 1 2 2 2)			S	2	17.93698634	302400
(1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 1 2)	A	D	S	2	18.62359538	151200
(1 1 1 1 1 2)		D		6	18.92894674	100800
(1 1 1 2 2 2 2 1 1 1 2 2 2 2)	A	D	S	2	19.28766287	151200
(1 1 1 1 1 1 1 1 1 1 1 1 2 2 1 2 2)		D		2	20.28162525	302400
(1 2 2 2)	A	D	S	7	20.28704612	43200
(2 2)	A	D	S	12	20.83206901	25200
(1 1 2 1 1 2 2 1 1 2 1 1 2 2 2 2)			S	2	21.17574254	302400
(1 1 1 1 2 1 1 1 1 2)	A	D	S	4	21.70718883	75600
(1 1 1 1 2 1 1 1 2 1 1 1 2 1 1 1 2 1 1 2)				2	21.81642193	604800
(1 2 2 2 2 2 2 1 2 2 2 2 2 2)	A	D	S	2	22.1106433	151200
(1 1 1 2 1 1 1 2 2 2 2 2 2 2 2 2)		D		2	22.59445537	302400
(1 2 2 2 1 2 2 2 2 2)		D		3	22.65918963	201600
(1 1 1 1 1 2 1 1 1 2 1 1 1 2 1 1 1 2 2 2)				2	22.69934531	604800
(1 1 1 1 2 1 1 1 2 2 2 1 2 2 1 1 2 2)				2	22.7315638	604800
(1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2)		D		2	22.76203997	302400
(1 1 1 2 2 1 1 2 2 1 1 1 2 2 1 1 2 2)	A	D	S	2	23.06301429	151200
(1 1 1 1 1 1 1 2 1 1 1 2 2 1 1 1 1 2 2 2)				2	23.43769594	604800
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2)				2	23.52009941	604800
(1 1 1 1 1 2 2 2 1 2 2 2)		D		3	23.52093941	201600
(1 1 1 1 2 1 2 2 2 2 2 1 1 2 1 1 2 2)				2	23.53126277	604800
(1 1 1 2 1 1 1 2 1 1 1 2 2 1 1 2 1 1 2 2)		D		2	23.60691475	302400
(1 1 1 2 1 1 2 1 1 1 2 1 1 2)	A	D	S	3	23.63884714	100800
(1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 2 2 2 2 2)		D		2	23.65096218	302400
(1 1 1 2 1 2 2 1 2 2 1 1 2 1 2 2 2 2)			S	2	23.67041023	302400
(1 1 1 1 1 1 1 2 1 1 1 2 2 1 1 2 2 2 1 2)				2	23.70785271	604800
(1 1 1 1 1 1 2 1 1 2 1 2 2 1 1 2 1 1 2 2)				2	23.89485923	604800

Table C.33. Length spectrum, genus 8589A

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 2)		D		2	16.59230915	360696
(1 1 1 1 2 2 1 2 2 2 2 2)				2	16.59230915	721392
(1 1 1 1 1 2 1 1 2 2 1 1 1 1 2 2)				2	18.94845013	721392
(1 1 1 2 1 1 2 2 2 2 1 2 2 2)				2	18.94845013	721392
(1 1 1 1 2 1 1 1 1 2 1 1 1 1 2 1 1 2)			S	2	19.83145703	360696
(1 1 1 1 1 1 1 1 2 1 1 1 2 2 1 1 1 2)			S	2	19.83145703	360696
(1 1 1 1 1 1 1 2 2 1 1 1 2 2 2 2)				2	19.83145703	721392
(1 1 1 1 2 2 1 1 1 1 2 2 1 1 2 2)			S	2	19.83145703	360696
(1 1 1 1 1 2 1 1 2 2 2 1 2 2 1 2)				2	19.83145703	721392
(1 1 2 1 2 2 2 2 2 1 1 2 2 2)			S	2	19.83145703	360696
(1 1 1 1 1 2 1 1 1 1 2 1 1 1 1 2 2 2)				2	20.71442478	721392
(1 1 1 1 1 1 2 1 2 2 2 2 1 2 2 2)				2	20.71442478	721392
(1 1 1 1 2 1 1 1 1 2 1 1 1 2 2 2 2 2)				2	22.18704417	721392
(1 1 1 1 1 1 1 1 2 1 2 2 1 1 2 2 2 2)				2	22.18704417	721392
(1 1 1 1 2 1 1 2 1 1 1 2 2 1 1 2 2 2)				2	22.18704417	721392
(1 1 1 1 2 1 1 2 2 2 1 1 2 1 1 1 2 2)				2	22.18704417	721392
(1 1 1 1 2 1 1 2 2 1 1 1 2 2 2 1 1 2)				2	22.18704417	721392
(1 1 1 1 1 2 1 2 2 1 1 2 2 1 2 2 1 2)				2	22.18704417	721392
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 2 1 1 2)				2	23.06996312	721392
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 2 1 1 1 1 2)				2	23.06996312	721392
(1 1 1 1 1 2 1 1 1 2 1 1 2 1 1 2 1 1 2 2)				2	23.06996312	721392
(1 1 1 1 1 2 2 1 1 1 1 2 2 1 2 2 2 2)				2	23.06996312	721392
(1 1 1 1 1 1 1 2 2 1 2 2 2 1 1 2 2 2)				2	23.06996312	721392
(1 1 1 1 2 1 1 2 2 1 1 2 2 2 2 1 1 2)				2	23.06996312	721392
(1 1 1 1 2 1 1 1 2 1 2 2 2 2 2 1 2 2)				2	23.06996312	721392
(1 1 1 2 1 2 2 1 2 2 1 1 1 2 2 1 2 2)				2	23.06996312	721392
(1 1 1 1 2 1 2 2 2 1 1 1 2 2 2 2 1 2)				2	23.06996312	721392
(1 1 2 2 1 2 2 1 2 2 1 2 2 2 2 2)				2	23.06996312	721392
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 1 1 1 1 1 1 2 2)				2	23.95287433	721392
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 2 2 1 1 2)				2	23.95287433	721392
(1 1 1 1 2 1 1 1 1 2 1 1 1 2 1 1 2 2 2 2)				2	23.95287433	721392
(1 1 1 1 1 1 2 1 1 1 2 2 1 1 1 2 2 1 2 2)				2	23.95287433	721392

Table C.34. Length spectrum, genus 8589B

Primitive translation	Symmetry			n	Length	#
(2 2)	A	D	S	8	13.88804601	45087
(1 1 1 1 1 1 1 2 1 1 2 2 2 2)				2	17.12948656	721392
(1 1 1 1 1 1 2 1 2 2 2 2 1 2)			S	2	17.12948656	360696
(1 1 1 1 1 1 2 1 1 1 2 1 1 1 2 1 1 2)				2	19.48546489	721392
(1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 1 2 2)				2	19.48546489	721392
(1 1 2 1 1 2 1 1 2 1 1 2 1 1 2 2)			S	2	19.48546489	360696
(1 1 1 1 2 2 2 2 1 1 2 2 2 2)			S	2	19.48546489	360696
(1 1 1 2 1 2 2 2 2 2 2 1 2 2)				2	19.48546489	721392
(1 1 1 1 1 1 1 2 1 1 1 1 1 2 2 2 1 2)				2	20.36844596	721392
(1 1 1 1 2 1 1 1 2 1 1 2 1 1 2 1 1 2)				2	20.36844596	721392
(1 1 1 1 2 1 1 1 2 2 1 1 2 2 2 2)				2	20.36844596	721392
(1 1 1 1 1 1 2 2 1 1 2 2 1 2 2 2)				2	20.36844596	721392
(1 1 2 1 2 2 2 2 2 1 2 2 2 2)				2	20.36844596	721392
(1 1 1 1 1 1 1 2 1 1 1 1 1 1 2 2 1 1 1 2)				2	21.84107785	721392
(1 1 1 1 2 1 1 2 1 1 2 1 1 2 1 2 2 2)				2	21.84107785	721392
(1 1 1 1 1 1 1 1 2 1 2 2 1 2 2 1 2 2)				2	21.84107785	721392
(1 1 1 2 1 1 2 1 1 2 1 1 1 2 2 1 2 2)		D		2	21.84107785	360696
(1 1 1 2 1 1 2 2 2 2 2 2 2 1 1 2)		D		2	21.84107785	360696
(1 1 1 1 1 1 1 2 1 1 1 1 1 2 2 1 1 1 2 2)				2	22.72400091	721392
(1 1 1 1 1 2 1 1 1 2 1 1 1 2 2 2 1 1 1 2)				2	22.72400091	721392
(1 1 1 1 2 2 1 1 1 2 1 1 2 1 2 2 2 2)				2	22.72400091	721392
(1 1 1 1 1 2 1 2 2 1 1 2 1 1 2 2 2 2)				2	22.72400091	721392
(1 1 1 1 1 1 1 1 2 1 2 2 2 2 2 1 2 2)				2	22.72400091	721392
(1 1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 1 2 1 1 2)				2	23.60691475	721392
(1 1 1 1 1 1 2 1 1 2 2 1 1 1 1 1 1 2 2 2)				2	23.60691475	721392
(1 1 1 1 1 1 1 1 2 1 1 1 1 2 2 2 1 1 2 2)				2	23.60691475	721392
(1 1 1 1 1 1 1 1 2 2 1 1 1 2 2 1 1 1 2 2)			S	2	23.60691475	360696
(1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 2 2 1 1 2)			S	2	23.60691475	360696
(1 1 1 1 1 2 1 1 2 1 1 1 2 1 2 2 1 1 2 2)				2	23.60691475	721392
(1 1 1 2 1 1 1 2 1 1 1 2 2 1 1 2 1 1 2 2)		D		2	23.60691475	360696
(1 1 1 1 1 1 1 1 1 2 1 2 2 1 1 2 2 2 1 2)				2	23.60691475	721392
(1 1 1 1 1 1 1 1 2 1 1 1 2 2 2 1 2 2 1 2)				2	23.60691475	721392

Table C.35. Length spectrum, genus 8589C

Primitive translation	Symmetry			n	Length	#
(1 1 1 1 1 1 1 1 1 1 1 1 2 2 2)		D		2	15.8021961	360696
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2)		D		2	18.15867163	360696
(1 1 1 1 1 2 1 1 1 1 1 2 1 2 2 1 2)				2	18.15867163	721392
(1 1 2 1 1 2 1 1 2 1 2 2 2 2)				2	18.15867163	721392
(1 1)	A	D	S	19	18.69574468	18984
(1 1 1 1 2 1 2 2)			S	4	19.04173162	180348
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2)			S	2	19.04173162	360696
(1 1 1 1 1 2 1 2 2 2 2 2 2 2)				2	19.04173162	721392
(1 1 1 2 2 2 1 1 2 2 1 2 2 2)				2	19.04173162	721392
(1 1 1 1 2 2 1 2 2 2 2 1 2 2)			S	2	19.04173162	360696
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2)			S	2	20.51443761	360696
(1 1 1 1 1 1 2 1 1 1 2 1 1 2 2 1 1 2)				2	20.51443761	721392
(1 1 1 1 1 2 1 1 2 1 1 2 1 1 2 1 1 2)		D		2	20.51443761	360696
(1 1 1 2 1 2 2 1 2 2 1 1 1 2 2 2)				2	20.51443761	721392
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 2 2)				2	21.39738496	721392
(1 1 1 1 1 1 2 1 1 1 1 1 2 2 1 2 2 2)				2	21.39738496	721392
(1 1 1 1 2 1 1 2 1 1 2 1 1 2 1 1 2 2)				2	21.39738496	721392
(1 1 1 1 1 2 1 1 2 2 1 1 2 1 1 1 2 2)				2	21.39738496	721392
(1 1 1 2 1 1 2 1 2 2 1 2 2 2 2 2)				2	21.39738496	721392
(1 1 1 2 1 2 2 1 2 2 1 1 2 2 2 2)				2	21.39738496	721392
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 2)				2	22.28031443	721392
(1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 2 1 1 2 2)				2	22.28031443	721392
(1 1 1 1 2 1 1 1 1 2 1 1 2 1 1 1 2 1 1 2)		D		2	22.28031443	360696
(1 1 1 1 1 2 2 1 2 2 1 1 1 1 1 2 2 2)		D		2	22.28031443	360696
(1 1 1 1 1 2 1 1 1 1 2 2 2 2 2 1 1 2)				2	22.28031443	721392
(1 1 1 2 1 1 2 2 1 1 2 2)				3	22.52010423	480928
(1 1 1 1 1 1 1 1 1 1 1 1 2 2 1 2 2 1 2 2)			S	2	23.50113952	360696
(1 1 2 1 1 2 1 2 2 2 1 1 2 2 1 1 2 2)			S	2	23.50113952	360696
(1 1 1 1 1 1 1 1 2 1 2 2 1 2)			S	3	23.5866594	240464
(1 1 1 2 2 1 1 2 2 1 2 2)				3	23.5866594	480928
(1 1 2 2 2 2 2 2 2 2)			S	3	23.5866594	240464
(1 1 1 1 1 1 1 2 1 1 2 1 1 1 1 1 1 1 2 1 1 2)	A	D	S	2	23.75289787	180348

Table C.36. Length spectrum, genus 11626

Primitive translation	Symmetry			n	Length	#
(1 2 2 1 2 2 2 2 2 2)			S	2	15.26796294	488250
(1 1 1 1 1 1 2 1 1 2 1 1 2 2 1 2)				2	18.50786215	976500
(1 1 1 2 2 1 1 2 1 1 2 2 2 2)				2	18.50786215	976500
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 2)				2	20.86357757	976500
(1 1 1 1 1 1 2 1 1 1 1 1 2 2 1 1 2 2)				2	20.86357757	976500
(1 1 1 1 1 1 1 1 2 2 1 1 1 2 1 1 2 2)				2	20.86357757	976500
(1 1 1 1 1 2 1 2 2 2 1 1 2 2 2 2)				2	20.86357757	976500
(1 1 1 1 2 1 1 1 2 2 2 2 1 2 2 2)				2	20.86357757	976500
(1 1 1 2 1 1 2 2 1 2 2 1 1 2 2 2)				2	20.86357757	976500
(1 1 1 1 2 1 2 2 2 2 2 1 1 1 2 2)				2	20.86357757	976500
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 1 1 2 2 1 2)				2	21.7465169	976500
(1 1 1 1 1 2 1 1 2 2 1 1 1 2 2 2 1 2)				2	21.7465169	976500
(1 1 1 1 1 2 2 1 2 2 1 2 2 2 2 2)				2	21.7465169	976500
(1 1 1 1 2 2 2 1 1 2 1 2 2 2 2 2)			S	2	21.7465169	488250
(1 1 2 1 1 2 2 1 2 2 2 1 1 2 2 2)				2	21.7465169	976500
(1 1 1 1 1 1 2 1 1 1 1 1 2 1 1 1 1 2 2 2)				2	22.6294412	976500
(1 1 1 1 1 2 1 1 1 2 1 1 2 1 1 1 2 2 1 2)				2	22.6294412	976500
(1 1 1 1 1 1 1 1 2 1 2 2 2 1 2 2 2 2)				2	22.6294412	976500
(1 1 1 1 2 1 1 2 1 2 2 2 1 1 1 2 2 2)				2	22.6294412	976500
(1 1 1 1 2 1 2 2 2 2 2 1 1 1 2 1 1 2 2)				2	22.6294412	976500
(1 1 1 1 1 1 2 1 1 2 2 2 2 1 2 2 1 2)				2	22.6294412	976500
(1 1 1 2 2 1 1 2 2 2 2 2 2 1 2 2)				2	22.6294412	976500
(1 1 1 1 1 1 1 1 1 1 1 2 2 1 1 2 1 1 2 2)		D		2	22.96735011	488250
(1 1 1 1 2 1 1 1 2 1 1 2 2 1 1 1 2 1 1 2)				2	22.96735011	976500
(1 1 1 1 1 2 1 1 1 2 2 2 1 1 2 2 2 2)				2	22.96735011	976500
(1 1 1 1 1 1 1 2 1 1 1 2 1 1 2 1 1 2 2 2)				2	23.2191096	976500
(1 1 1 1 1 2 1 1 1 1 2 1 1 1 1 2 2 1 2 2)				2	23.2191096	976500
(1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 2 2 2 1 2)				2	23.2191096	976500
(1 1 1 1 1 1 2 1 1 2 2 1 1 2 2 2 2 2)				2	23.2191096	976500
(1 1 1 2 1 1 2 2 2 1 1 1 2 1 2 2 2 2)				2	23.2191096	976500
(1 1 1 2 1 1 1 2 1 1 2 2 2 2 1 2 2 2)				2	23.2191096	976500
(1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 1 2 2 1 2)				2	24.1020198	976500

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BIOGRAPHICAL SKETCH

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The author grew up with an inexplicable love of math and science in the small city of Elko, Nevada (which, during his youth, National Geographic magazine called 'the last cow town in the West'). After leaving home, he served time intermittently at various respectable institutions, where he was known at times as a biology major and a physics major. Due largely to the influence of a few outstanding math teachers, he eventually left the University of Utah with two degrees in mathematics. The next several years were spent teaching math, most memorably at Hollenbeck Middle School in East Los Angeles and most enjoyably at Salt Lake Community College. His life then took an abrupt turn when he quite unexpectedly found himself deciding to go back to school and was graciously offered a visiting position at Brigham Young University in order to begin working toward a doctorate. More surprises were in store, and the author's family soon found themselves packing up and moving to Tallahassee in order to study under Phil Bowers at Florida State University. The result, of course, is the present dissertation and the upcoming graduation. At the moment of this writing, the author has just arrived in Finland where he has accepted a research position at the University of Helsinki's Graduate School in Analysis.