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Solving Linear Differential Equations in Terms of Hypergeometric Functions by #-Descent

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SOLVING LINEAR DIFFERENTIAL EQUATIONS IN TERMS OF HYPERGEOMETRIC FUNCTIONS BY 2-DESCENT

By

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LIST OF SYMBOLS

The following short list of symbols are used throughout this thesis.

K	Differential field, most time represents $\mathbb{C}(x)$
k	Subfield of K
C_K	The constant field of K
C	Ground field of the involved differential equation
∂	Derivation $\frac{d}{dx}$
$K[\partial]$	Ring of differential operator defined over K
L	Differential operator in $K[\partial]$
L^*	Adjoint operator of L
$V(L)$	Solution space of the differential operator L
$\mathcal{D}$	$K[\partial]$
$\otimes$	Tensor product of two modules
$\sim_g$	Gauge equivalence between two differential operators
$\sim_p$	Projective equivalence between two differential operators
Ⓢ	The symmetric product sign of two differential operators
${}_pF_q$	The generalized hypergeometric series
$\Delta(L, p)$	The exponent-difference of L at point p

ABSTRACT

Let L be a linear ordinary differential equation with coefficients in $\mathbb{C}(x)$. This thesis presents algorithms to solve L in closed form. The key part of this thesis is 2-descent method, which is used to reduce L to an equation that is easier to solve. The starting point is an irreducible L, and the goal of 2-descent is to decide if L is projectively equivalent to another equation $\tilde{L}$ that is defined over a subfield $\mathbb{C}(f)$ of $\mathbb{C}(x)$.

Although part of the mathematics for 2-descent has already been treated before, a complete implementation could not be given because it involved a step for which we do not have a complete implementation. Our key novelty is to give an approach that is fully implementable. We describe and implement the algorithm for order 2, and show by examples that the same also work for higher order. By doing 2-descent for L, the number of true singularities drops to at most $n/2 + 2$ (n is the number of true singularities of L). This provides us ways to solve L in closed form(e.g.in terms of hypergeometric funtions).

CHAPTER 1

INTRODUCTION

Let $L = \sum_{i=0}^{n} a_i \partial^i$ be a differential operator with coefficients a_i in a differential field $K = \mathbb{C}(x)$, where ∂ is the usual differentiation $\frac{d}{dx}$. The corresponding differential equation is $L(y) = 0$, i.e. $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$. Finding the closed form solutions of L in computer algebra systems is important because of its wide applications in Physics, Combinatorics and other fields. Algorithms that can be fully implemented in computer algebra systems are also needed.

1.1 Closed Form Solutions of L

When we say a linear ordinary differential equation has closed form solutions, we mean that solutions can be written in terms of functions from a defined set of functions, under operations from a defined set of operations. In [22], these functions are $\{\mathbb{C}(x)$, exp, log, Airy, Bessel, Kummer, Whittaker, and ${}_2F_1$-Hypergeometric functions $\}$ and the operations are $\{$field operations, algebraic extensions, compositions, differentiation and $\int dx\}$. Solving a second order L in terms of these functions is solved in [30] except for ${}_2F_1$-Hypergeometric Functions. Thus, our focus will be on finding ${}_2F_1$ type solutions. Differential equations with ${}_2F_1$-type solutions are very common in Combinatorics and Physics [2, 5]. However, there are no complete algorithms to find such solutions.

Example 1.

$$L := \partial^2 + \frac{4(1296x^5 + 576x^4 - 144x^3 - 72x^2 + x + 1)}{x(6x-1)(2x+1)(6x+1)(12x^2-1)}\partial + \frac{2(5184x^6 - 864x^5 - 1656x^4 + 48x^3 + 162x^2 + 6x - 1)}{(-1+2x)x^2(6x-1)(2x+1)(6x+1)(12x^2-1)}$$

This differential equation is satisfied by a D-finite generating function that counts certain lattice walks in the quarter plane [27, 5]. However, current computer algebra system does not solve it. Of the 17 equations in [27, 5], there are 16 such equations that can not be solved by current computer algebra system (despite the fact that they do have closed form solutions). Therefore it becomes important to develop algorithms to solve such equations, especially algorithms to find the ${}_2F_1$-type solutions. We will continue Example 1 in Section 3.7 and Section 5.4.

1.2 Descent method for differential equations

The problem of finding closed form solutions of L becomes easier if we can factor L as a product of lower order operators as in [3, 6, 17] or apply some other approach to reduce the order, see [19, 28].

A different type of reduction is called *descent*. The goal is to reduce L to an operator $\tilde{L}$ of the same order, but this time defined over a proper subfield $k = \mathbb{C}(f)$ of K. Here $\tilde{L}$ must be *projectively equivalent* to L. Informally, this means that L can be solved in terms of the solutions of $\tilde{L}$ and vice versa (a precise definition will be given later in Chapter 2).

In this thesis, we treat of 2-descent, meaning that k is a subfield of K with index 2. We focus on solving second order equations in terms of hypergeometric functions by 2-descent, at the end we will also give examples for higher order. For a second order equation L, after applying Kovacic' algorithm [26], we can assume that L is irreducible (i.e. not a product of lower order factors), and that it has no Liouvillian solutions.

Descent reduces the number of true singularities (Definition 18) from n to at most $n/2+2$, which helps to solve differential equations as illustrated in Section 3.7 and Section 4.3. In particular, for second order equations, if the number of true singularities[1] drops to 3, and if these are regular singularities[2], then a ${}_2F_1$-type solution can be obtained quickly. We can also stop reducing when we reach a second order operator with four true singularities, because 4-singularity equations with ${}_2F_1$-type solutions are currently being classified by [16]. Classifying equations with closed form solutions and > 4 singularities would be hard to do, this is where 2-descent becomes crucial.

[1] the number of *removable* singularities (Def. 18) is irrelevant

[2] for the irregular singular case, finding closed form solutions if they exist can be done with [12, 23]

If $L \in \mathbb{C}(x)[\partial]$ then there is a finitely generated extension $\mathbb{Q} \subseteq C$ with $L \in C(x)[\partial]$, just take C to be the extension of $\mathbb{Q}$ given by the coefficients of L. The main design goal for our algorithm is to introduce as few algebraic extensions of C as possible. Without this design goal, Sections 3.3 and 3.5 would have been much shorter (if we simply compute the splitting field of the singularities then for Section 3.5 we can follow [8] and Section 3.3 becomes trivial. Sections 3.3 and 3.5 become non-trivial when we aim to minimize field extensions).

The main results in this thesis is the 2-descent algorithm in Chapter 3 and Chapter 4. We know from [21] that if there is a gauge transformation G from L to $\sigma(L)$, then L will allow descent with respect to σ. The question is, given G, how to find the descent? Is it necessary (as in the terminology in [21]) to trivialize a 2-cocycle, or to perform some equivalent complicated operation such as finding a point on a conic over $C(x)$? The answer is no; we give a short and efficient algorithm in Section 3.4 (and an alternative method in Chapter 4), and we even show (Theorem 1) that it produces a result over an optimal extension of C.

1.3 Relation to prior work

For a second order differential equation, it is shown in [8, 21] that the problem of computing 2-descent can be reduced to another problem (trivializing a 2-cocycle) although no step by step algorithm is given in these papers. The paper [19] does give an algorithm, and implementation, that can be used to find 2-descent, as follows. If σ is a Möbius transformation of order 2, and $\mathbb{C}(f)$ is the fixed field of σ, and if L is projectively equivalent to $\sigma(L)$, then we can compute the so-called symmetric product of $L, \sigma(L)$, then apply factorization (DFactorLCLM in Maple), take the 3'rd order factor found that way, and run the algorithm from [19] to find a second order operator. All of these steps are implemented, and the end result is a 2-descent.

The problem with the above methods is that they rely on an algorithm that can find a point on a conic defined over K (or an algorithm that solves an equivalent problem). Although such a point must exist when $K = \mathbb{C}(x)$, the proof does not show how to find such a point over a field of constants that is optimal or close to optimal (recall that we wish to minimize the extension of C that the algorithm introduces, where $C \subset \mathbb{C}$). There is only an implementation in [20] for this step if C is $\mathbb{Q}$ or a transcendental extension of $\mathbb{Q}$. If L contains algebraic numbers, then there is no implementation for finding a point on a conic, and without that, it is not clear how to obtain from [19, 8, 21], a complete implementation for finding 2-descent.

In Chapter 3 of this thesis, a step by step algorithm is described for finding 2-descent for a second order differential equation. The algorithm can be fully implemented [13] because it does not call a conic algorithm. Note: If $L \in C(x)[\partial]$ with $C \subset \mathbb{C}$ of order 2, and if one allows unnecessary algebraic extensions of C (potentially exponentially large), then it is not hard to implement a conic algorithm, in which case one can consider 2-descent an already solved problem. But in practice our algorithm would be much preferable because it only extends C when necessary (i.e. when there is no 2-descent defined over C).

This thesis is organized as follows: Chapter 2 contains the preliminary knowledges of differential equations and differential modules. In Chapter 3, the 2-descent algorithm is described in details. Several support theorems are also proved. Chapter 4 presents an improved algorithm for the *Case A* of finding 2-descent. We also give examples to show the application of 2-descent for higher order differential equations in Chapter 4. Chapter 5 shows how to solve second order linear differential equations in terms of Hypergeometric functions by examples. We give the conclusion in Chapter 6.

CHAPTER 2

SINGULARITIES AND TRANSFORMATIONS

In this chapter, we first introduce some facts about differential equations, differential operators, and their singularities. After that, the transformations between differential operators will be discussed. Because these properties are already known, we will just list them, and refer to [11, 23, 29] for proofs and details.

2.1 Differential Operator Ring

Definition 1. *([29])*

1. *An ordinary differential ring R is a ring R equipped with a map (derivation) $\partial : R \longrightarrow R$, such that:*

$$\partial(a+b) = \partial(a) + \partial(b),$$

$$\partial(ab) = \partial(a)b + a\partial(b),$$

   for $a, b \in R$

2. *The ring $C_R = \{c \in R | \partial(c) = 0\}$ is called the ring of constants of R.*

Remark 1. *An ordinary differential ring R which is also a field is called a differential field. If K is a differential field, the corresponding ring of constants is a field C_K, as can be proved directly from the definition.*

Example 2. *The ring of formal power series $\mathbb{Z}[[x]]$ with derivation $f' = \frac{df}{dx}$ is a differential ring, the ring of constants of which is $\mathbb{Z}$.*

Example 3. *The field of rational functions $\mathbb{C}(x)$ with derivation $f' = \frac{df}{dx}$ is a differential field, the field of constants of which is $\mathbb{C}$.*

Note: In this thesis, the coefficients of the differential equations are from a differential field K, where K is $\mathbb{C}(x)$ or a subfield of $\mathbb{C}(x)$.

Definition 2. *Let K be a differential field. The ring of non-commutative polynomials of the form $L = a_n\partial^n + \cdots + a_1\partial + a_0$ with $a_i \in K$ is called the ring of differential operators over K, where the ring multiplication is defined by $\partial \cdot a = a^{'} + a\partial$ for all $a \in K$. An operator L acts on functions as $L(y) = a_n y^{(n)} + \cdots + a_1 y' + a_0 y$.*

Remark 2. *Multiplication of differential operators is defined in such a way that: $L_1(L_2(y)) = (L_1L_2)(y)$. Through this thesis, we denote this differential operator ring as $\mathcal{D}= K[\partial]$. The equation $L(y) = 0$ is same as the scalar differential equation $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$. The order of $L = a_n\partial^n + \cdots + a_1\partial + a_0$ is n if $a_n \neq 0$. We use differential operators and differential equations interchangeably.*

Lemma 1. *Every left ideal of $\mathcal{D}$ is of the form $\mathcal{D}L$ for some $L \in \mathcal{D}$.*

Sketch of proof: $\mathcal{D}$ has all the properties of a Euclidean ring, except commutativity. It has right-division and left-division. Using right-division, one can compute the greatest common right divisor (GCRD). In other words, the Euclidean algorithm works for $\mathcal{D}$, and the lemma then follows.

Remark 3. *We can also define the Least Common Left Multiple of $L_1, L_2 \in \mathcal{D}$ (Denote as LCLM(L_1, L_2)) as the unique monic generator of $\mathcal{D}L_1 \cap \mathcal{D}L_2$ [31].*

Let Ω be a universal extension of $\mathbb{C}(x)$ as in [29], which is a $\mathcal{D}$-module as well as a commutative ring with the property $\dim_{\mathbb{C}}(\ker(L, \Omega \to \Omega))=\mathrm{ord}(L)$ for any $L \in \mathcal{D}$-$\{0\}$. Then we have the following definition.

Definition 3. *Given a linear differential operator $L \in \mathcal{D}$ of order n, the solution space $V(L)$ of $L(y) = 0$ in K is defined as $\ker(L, \Omega \to \Omega)$. Then $V(L)$ is a vector space over $\mathbb{C}$ of dimension $\mathrm{ord}(L)$.*

Lemma 2. *Given $L_1, L_2 \in \mathbb{C}(x)$, then $V(GCRD(L_1, L_2)) = V(LCLM(L_1, L_2))$.*

The next lemma gives the relation between two differential operators when they have the same solution spaces.

Lemma 3. *If $L_1, L_2 \in \mathbb{C}(x)[\partial]$ have the same order, then*

$$V(L_1) = V(L_2) \Longleftrightarrow L_1 = \frac{\mathrm{lc}(L_1)}{\mathrm{lc}(L_2)} L_2$$

where lc stands for leading coefficient $\mathrm{lc}(a_n \partial^n + \cdots + a_1 \partial + a_0) = a_n$ *if* $a_n \neq 0$.

Proof. $'' \Rightarrow ''$: This can be verified by Lemma 2 that $V(\mathrm{GCRD}(L_1, L_2)) = V(L_1) \cap V(L_2) = V(L_1) = V(L_2)$ and $\mathrm{GCRD}(L_1, L_2)$ is a right factor of both L_1 and L_2. □

2.2 Differential Module

A linear differential equation can be presented in several ways: scalar form, that is the form we usually adopt; matrix form; differential module form. These three forms are equivalent to each other, we can use these three forms interchangeably [29].

Definition 4. *A differential module M is a finite dimensional K-vector space equipped with an additive map $\partial : M \longrightarrow M$ with the property: $\partial(fm) = f'm + f\partial m$ for all $f \in K$ and $m \in M$. In other words, M is a $\mathcal{D}$-module, and is finite dimensional as K-vector space.*

We denote a differential module as (M, ∂).

Definition 5. *Let (M_1, ∂_1) and (M_2, ∂_2) be two differential modules. A differential module homomorphism $\phi : M_1 \to M_2$ is a K-linear map such that $\phi(\partial_1(m)) = \partial_2(\phi(m))$ for all $m \in M_1$.*

Definition 6. *(M_1, ∂_1) and (M_2, ∂_2) are isomorphic if there exists a bijective differential homomorphism $\phi : M_1 \to M_2$.*

Definition 7. *Let M be a differential module over K. An element $e \in M$ is called a cyclic vector if M is generated over K by the elements $e, \partial e, \partial^2 e, \ldots$ (i.e. M is generated over $\mathcal{D}$ by e).*

Lemma 4. *Assume $K \neq C_K$. Let M be a differential module with K-basis $\{e_1, \cdots, e_n\}$ and let $\eta_1, \cdots, \eta_n \in K$ be a linearly independent over C_K. Then there exist integers $0 \leq c_{i,j} \leq n$, $1 \leq i, j \leq n$, such that $m = \sum_{i=1}^{n} a_i e_i$ is a cyclic vector for M, where $a_i = \sum_{j=1}^{n} c_{i,j} \eta_j$.*

Proof. See [26, 29]. □

Corollary 1. *Assume $K \neq C_K$. Every differential module is isomorphic to a module of the form $\mathcal{D}/\mathcal{D}L$ for some $L \in \mathcal{D}$.*

Proof. Assume e is a cyclic vector in M by Lemma 4, we define the $\mathcal{D}$-module homomorphism $\phi : \mathcal{D} \mapsto M$, which is defined by $e = \phi(1)$, where $1 \in \mathcal{D}$ is the unit element. Then ϕ is surjective because e is a cyclic vector in M, furthermore, we know its kernel is a left ideal and has the form $\mathcal{D}L$ for some $L \in \mathcal{D}$. Therefore $M \cong \mathcal{D}/\mathcal{D}L$. □

This way, we associate a scalar differential operator L over K to a differential module M.

Remark 4. *Note that L is not unique in Corollary 1; A differential module M has many cyclic vectors. Taking two cyclic vectors, we obtain two operators L_1 and L_2 such that $M \cong \mathcal{D}/\mathcal{D}L_1$ and $M \cong \mathcal{D}/\mathcal{D}L_2$. The property $\mathcal{D}/\mathcal{D}L_1 \cong \mathcal{D}/\mathcal{D}L_2$ is thus important; we say L_1 and L_2 are* gauge equivalent *when $\mathcal{D}/\mathcal{D}L_1 \cong \mathcal{D}/\mathcal{D}L_2$.*

2.3 Singularities of L

In this thesis, the singularities of differential equations play a significant role. Therefore, we discuss here some properties of singularities and their roles in finding solutions.

Definition 8. *A point $p \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is called a singularity of a differential operator $L \in K[\partial]$, if p is a zero of the leading coefficient of L or p is a pole of one of the other coefficients of L. p is called a* regular point *if it is not a singularity.*

Example 4. *Let*

$$L = \partial^2 + \frac{28x-5}{x(4x-1)}\partial + \frac{144x^2+20x-3}{x^2(4x-1)(4x+1)}.$$

To find the singularities of L, we first find the zeros of leading coefficient, since the leading coefficient is 1, so no zeros for this part. Next, we compute the poles for the other coefficients, which are the set $\{0, \frac{1}{4}, -\frac{1}{4}\}$. However, we are not done at this point, we still need consider the behavior of the point $x = \infty$. To examine this, we should do a change of variable $x \mapsto \frac{1}{x}$, in this way, ∞ becomes 0 and we only need examine if $x = 0$ is a singularity in this new differential operator. After checking, we find $x = 0$ is a singularity, that means ∞ should be included in the singularities set of L. Therefore, we find the singularities set of L, which is $\{0, \infty, \frac{1}{4}, -\frac{1}{4}\}$.

Remark 5. *(Cauchy's Theorem) If* p *is a regular point of* L*, we can write all solutions of* L *at* p *as convergent power series* $\sum_{i=0}^{\infty} a_i t_p^i$*, where* t_p *denotes the local parameter which is* $t_p = \frac{1}{x}$ *if* $p = \infty$ *and* $t_p = x - p$*, otherwise.*

Among the singularities of a given differential operator L, we classify them into several classes. For convenience, we suppose our differential operator L to be monic, that means the leading coefficient is 1.

Definition 9. *[30] A singularity* p *of* L *is:*

1. *regular singularity* $(p \neq \infty)$ *if* $t_p^i a_{n-i}$ *is analytic at* $x = p$ *for* $1 \leq i \leq n$*.*
2. *regular singularity* $(p = \infty)$ *if* $\frac{a_{n-i}}{t_\infty^i}$ *is analytic at* $x = p$ *for* $1 \leq i \leq n$*.*
3. *irregular singularity otherwise.*

For the 2-descent method we discuss later, the exponents at a singular point of a differential operator play an important role.

Definition 10. *Given a differential operator* $L \in K[\partial]$*, an element* $e \in \mathbb{C}[[t_p^{-1/r}]]$*,* $r \in \mathbb{N}$ *is called a* generalized exponent *of* L *at the point* p *if there exists a formal solution of the form*

$$y(x) = \exp\left(\int \frac{e}{t_p}\, dt_p\right) S \tag{2.1}$$

where $S \in \mathbb{C}[[t_p^{1/r}]][\ln(t_p)]$*, and the* t_p^0*-term of* S *is non-zero. If* $e \in \mathbb{C}$ *then (2.1) simplifies to* $t_p^e S$*, in which case* e *is an* exponent *of* L*.*

Remark 6. *Given a differential operator* L*, we can compute the generalized exponents at a point* p *by the* ***Maple*** *command gen_exp, we will give more details in the appendix.*

Theorem 1. *Given a differential operator* $L \in \mathcal{D}$ *of order* n*, suppose that the ramification indices of the generalized exponents divide* $r \in \mathbb{N}$*. Then there exists a basis* $\{y_1, \cdots, y_n\}$ *of* $V(L)$*, of the form*

$$y_i(x) = \exp\left(\int \frac{e_i}{t_p}\, dt_p\right) S_i,$$

with $S_i \in \mathbb{C}[[t_p^{1/r}]][\ln(t_p)]$*, where* $e_1, \cdots, e_n \in \mathbb{C}[[t_p^{-1/r}]]$ *are generalized exponents and the* t_p^0*-term of* S_i *is non-zero.*

Proof. See [17]. □

For order 2, we give some more details:

Theorem 2. *Suppose L is a second order linear differential operator and p is a singularity, then:*

1. *If p is a regular singular point of L, then there exist two linearly independent solutions*

$$y_1(x) = t_p^{e_1} \sum_{i=0}^{\infty} a_i t_p^i,\ \ a_0 \neq 0$$

and

$$y_2(x) = t_p^{e_2} \sum_{i=0}^{\infty} b_i t_p^i + c y_1(x) \ln(t_p),$$

where $e_1, e_2, a_i, b_i, c \in \overline{C}$ and b_0 and c are not both 0.

2. *If p is an irregular singular point of L, then the two linearly independent solutions are*

$$y_1(x) = \exp(\int \frac{e_1}{t_p} dt_p) \sum_{i=0}^{\infty} a_i t_p^{i/r},\ \ a_0 \neq 0$$

and

$$y_2(x) = \exp(\int \frac{e_2}{t_p} dt_p) \sum_{i=0}^{\infty} b_i t_p^{-i/r} + c y_1(x) \ln(t_p),$$

where $a_i, b_i, c \in \overline{C}$, $e_1, e_2 \in \mathbb{C}[t_p^{-1/r}]$ and b_0 and c are not both 0.

Proof. See [24, 30]. □

Remark 7. *In Theorem 2, if $e_1 - e_2 \notin \mathbb{Z}$, then $c = 0$ and y_2 does not include a logarithmic term. If $c \neq 0$, we say L has logarithmic solutions at $x = p$.*

For a given second order linear differential operator $L := \partial^2 + P(x)\partial + Q(x)$, if $x = p$ is a regular singular point, then the exponents e_1, e_2 can be determined by solving the corresponding indicial equation:

$$\lambda(\lambda - 1) + p_0 \lambda + q_0 = 0$$

where p_0 and q_0 are the constant coefficient of the power series expansion of $(x-p)P(x)$ and $(x-p)^2 Q(x)$ at the point p, respectively.

2.4 Transformations

There are three known types of transformations that send, for any n'th order $L_1 \in K[\partial]$, the solution space of L_1 to the solution space of some $L_2 \in K[\partial]$, again of order n. They are (notation as in [12])

1. change of variables: $y(x) \mapsto y(f(x))$, $\quad f(x) \in K \setminus C_K$.
2. exp-product: $y \mapsto e^{\int r\,dx} \cdot y$, $\quad r \in K$.
3. gauge transformation: $y \mapsto r_0 y + r_1 y' + \cdots + r_{n-1} y^{(n-1)}$, $\quad r_0, r_1 \cdots, r_{n-1} \in K$.

Definition 11. *Let $L_1, L_2 \in K[\partial]$. They are called gauge equivalent (notation: $L_1 \sim_g L_2$) if there exists a so-called gauge transformation, which means a bijection from $V(L_1)$ to $V(L_2)$ of the form in item 3.*

Remark 8. *Let L_1, $L_2 \in K[\partial]$. The $\mathcal{D}$-modules $\mathcal{D}/\mathcal{D}L_i$, $i = 1, 2$ are isomorphic if and only if $L_1 \sim_g L_2$. In particular, $\sim_g$ is an equivalence relation (see [3]).*

Definition 12. *Let L_1, $L_2 \in K[\partial]$. They are called* projectively equivalent *(notation: $L_1 \sim_p L_2$) if there exists a bijection $V(L_1) \to V(L_2)$ of the form*

$$y \longrightarrow e^{\int r} \cdot (r_0 y + r_1 y' + \cdots + r_{n-1} y^{(n-1)}) \tag{2.2}$$

for r, $r_0, \cdots, r_{n-1} \in K$ (i.e. a combination of item 2 and item 3).

Remark 9. *[21] Let L_1, $L_2 \in K[\partial]$. $L_1 \sim_p L_2$ if and only if there exists a $\mathcal{D}$-module E of dimension 1 over K such that $\mathcal{D}/\mathcal{D}L_1 \cong E \otimes \mathcal{D}/\mathcal{D}L_2$.*

Remark 10. *Projective equivalence is also an equivalence relation, the details are in [3]. A projective equivalence relation between two differential operators L_1, L_2 is important for solving because it means we can solve L_1 in terms of the solutions of L_2, and vice versa.*

The algorithm for finding (if it exists) a projective equivalence between two given n'th order differential operators is also given in [3]. An implementation can be found in Maple package ISOLDE. Through this thesis, we will frequently decide whether two second order linear differential operator are projectively equivalence, a typical faster algorithm **equiv** for order 2 is developed and implemented in [18]. We use this algorithm to decide if $L_1 \sim_p L_2$, and if so, to find the projective equivalence (the r, r_0, r_1 in (2.2)). A summary of this algorithm will be given in the remainder of this chapter.

Theorem 3. *Let $L_1, L_2 \in K[\partial]$ of order 2, the question of finding the projective equivalence between L_1 and L_2 can be reduced to find a hyperexponential solution of a system of linear differential equations.*

Proof. The proof of this theorem is reproduced from [11]. The proof also gives the idea of how this algorithm works.

If $L_1 \sim_p L_2$, then there is an operator $G = \exp(\int r)G_1$ such that L_1 is a right factor of L_2G, where $r \in K$ and $G_1 \in K[\partial]$ with order one. Set $G = r_1\partial + r_0$, where $r_0 = \exp(\int r)s_0$ and $r_1 = \exp(\int r)s_1$. In this case, we need find two hyperexponential functions r_0 and r_1. Since L_2G is right divisible by L_1, therefore the remainder R should be 0. In general, R is an operator of order one and its coefficients are K-linear combinations of $r_0, r_0', r_0'', r_1, r_1'$ and r_1''. Equating these coefficients to 0 yields a system of two differential equations of order two with two unknowns r_0, r_1.

After replacing r_0' and r_1' by another two variables r_2 and r_3, and adding the equations $r_0' - r_2 = 0$ and $r_1' - r_3 = 0$, we get a new system of differential equations of order 1 with 4 unknowns. Therefore, finding the projective equivalence between L_1 and L_2 is reduced to finding hyperexponential solutions r_0, r_1, r_2, r_3 of a system of differential equations. □

Before giving the procedures of finding hyperexponential solutions of a system of differential equations, it is necessary to define the *adjoint* of a differential operator L.

Definition 13. *Given a differential operator $L := a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0$, the adjoint of L is defined as $L^* := \sum_{j=0}^{n}(-1)^j\partial^j a_j$.*

Remark 11. *For two differential operator L_1, L_2, we have $(L_1L_2)^* = L_2^*L_1^*$ and $L_1^{**} = L_1$. This fact can be verified easily.*

The *cyclic vector* method is used to find the hyperexponential solutions of a system of differential equations. The proof of this method can be found in [11].

Algorithm: Finding the projective equivalence between L_1 and L_2.

Input: L_1, L_2 of order 2.

Output: G, if $L_1 \sim_p L_2$ under G.

Step 1: Write $G = r_1\partial + r_0$ with $r_0 = \exp(\int r)s_0$ and $r_1 = \exp(\int r)s_1$ for some $r, s_0, s_1 \in K$.

Step 2: L_2G should be divided by L_1, so the remainder R equals to 0. This produces a system of differential equations of $r_0, r_0', r_0'', r_1, r_1'$ and r_1''.

Step 3: Set $r_2 = r_0'$ and $r_3 = r_1'$, we get another system of first order differential equations with variables r_0, r_1, r_2, r_3.

Step 4: Pick a random $v \in K^4$.

Step 5: Check whether v is cyclic, otherwise, redo step 5 and step 6.

Step 6: Compute $L = a_0 + a_1\partial + a_2\partial^2 + a_3\partial^3 + \partial^4$ such that $Lv = 0$.

Step 7: Compute a hyperexponential solution s of L^*. (By using the DEtools[expsols] command in Maple)

Step 8: Compute Q such that $L = (\partial + \frac{s'}{s})(\frac{1}{s}Q)$.

Step 9: Let $Q = y_0 + y_1\partial + y_2\partial^2 + y_3\partial^3$ and $y = y_0v + y_1\partial v + y_2\partial^2 v + y_3\partial^3 v$. Then y would be the hyperexponential solution.

Step 10: Take the first two parts of y as r_0 and r_1 respectively.

Example 5. *Consider the following two differential operators L_1 and L_2:*

$$L_1 := 7056x(x-1)\partial^2 - (-8232x + 3528)\partial + 13$$

$$L_2 := 7056x(x-1)\partial^2 + (1176x + 3528)\partial + 1189$$

```
>equiv(L_1,L_2);
```

$$G := \sqrt{x}((x-1)\partial + \frac{13}{3528})$$

Similarly, we can compute the equivalence from L_2 to L_1.

```
>equiv(L_2,L_1);
```

$$G := \frac{1}{\sqrt{x}}((x-1)\partial - \frac{1189}{3528})$$

CHAPTER 3

2-DESCENT REDUCTION FOR LINEAR DIFFERENTIAL OPERATORS

To solve a differential equation, the first way coming to mind is direct solving, i.e. comparing with textbook equations, and other existing techniques. What should we do if no solution was found? The next strategy we may consider is trying to reduce this equation to lower order. This is natural because it is usual easier to solve a lower order equation. Many such reduction method were developed in [26, 31, 19, 17]. What to do when the order can not be reduced? Another kind of reduction called *descent*. This kind of reduction also aims to reduce our differential equation to one which is easier to solve. In the rest part of this chapter, the 2-descent method will be developed.

3.1 Introduction of descent method

Definition 14. *Let K/k be an extension of differential fields and M be a differential module over K, we say that M i-descends (i means isomorphism) to k if there exists a differential module N over k such that M is isomorphic to $K \otimes_k N$.*

Remark 12. *From the view of differential operators, we say that L i-descends to k if there exists another differential operator $\tilde{L}$ over k such that L is gauge equivalent to $\tilde{L}$.*

Example 6. *Consider*

$$L = \partial^2 - \frac{1}{x}\partial - 4x^4 \in K[\partial]$$

where $K = \mathbb{C}(x)$.

Now let $f = x^2$ and consider $k = \mathbb{C}(f)$. In this example, it is easy to find a descent for

L to k because $V(L) = V(\tilde{L})$ for some $\tilde{L} \in k[\partial_f]$, namely (see also Lemma 3): less trivial examples occur when L descents to some $\tilde{L}$ with $V(L) \neq V(\tilde{L})$ (i.e. a nontrivial gauge transformation is needed.)

Comparing with i-descent, we have the definition of p-descent (p represents projective):

Definition 15. *Let K/k be an extension of differential fields and M be a differential module over K, we say that M p-descends to k if there exist a differential module N over k and an 1-dimension module E such that $E \otimes M$ i-descents to $K \otimes_k N$.*

Remark 13. *In most content descent means i-descent. Through this thesis, we consider more general descent: p-descent. That means we find a differential operator $\tilde{L}$ over k such that L is projectively equivalent to $\tilde{L}$. The algorithm in Section 3.4 will treat i-descent, the algorithm in Section 3.5 will find a 1-dimension module E that reduces p-descent to i-descent.*

If L descends to $\tilde{L}$ over a subfield k of K, we do not reduce the order of L, however, $\tilde{L}$ is easier to solve. More precisely:

- $\text{order}(L) = \text{order}(\tilde{L})$.
- $\tilde{L}$ is defined over k and k is a subfield of K.
- The number of singularities of $\tilde{L}$ is no more than that of L's.

In this thesis, we will focus on developing 2-descent method, which means p-descent to a subfield of index 2.

3.2 2-descent

Definition 16. *Let $f = \frac{A}{B}$ with $A, B \in \mathbb{C}[x]$ coprime, then the degree of f is defined as*

$$\deg(f) = \max(\deg(A), \deg(B)) = [\mathbb{C}(x) : \mathbb{C}(f)].$$

Remark 14. *If $\sigma \in \mathrm{Aut}(\mathbb{C}(x)/\mathbb{C})$ has order 2, then the fixed field of σ is a subfield of $\mathbb{C}(x)$ of index 2, and by Lüroth's theorem this subfield is of the form $\mathbb{C}(f)$, for some $f \in \mathbb{C}(x)$ of degree 2 (note: we can find such f in $\{x + \sigma(x), x\sigma(x)\} \setminus C$). Any subfield $\mathbb{C}(f) \subset \mathbb{C}(x)$ of* *index 2 is the fixed field of some $\sigma \in \mathrm{Aut}(\mathbb{C}(x)/\mathbb{C})$ of order 2 (after all, every extension of degree 2 is Galois). The automorphisms of $\mathbb{C}(x)$ over $\mathbb{C}$ are Möbius transformations:*

$$x \mapsto \frac{ax+b}{cx+d} \tag{3.1}$$

This chapter treats 2-descent, so we only consider σ of order 2, which is equivalent to having $d = -a$ in (3.1).

Remark 15. *Any $\sigma \in \mathrm{Aut}(\mathbb{C}(x)/\mathbb{C})$ extends to an automorphism of $\mathbb{C}(x)[\partial]$. If σ has finite order, and if $\mathbb{C}(f)$ is the fixed field of σ, and if $L \in \mathbb{C}(x)[\partial]$, then*

$$L = \sigma(L) \iff L \in \mathbb{C}(f)[\partial_f], \tag{3.2}$$

in other words, $\mathbb{C}(f)[\partial_f]$ is the fixed ring of σ. Here $\partial_f := \frac{d}{df} = \frac{1}{f'}\partial$, where $'$ is differentiation w.r.t. x.

Definition 17. *Let $L \in \mathbb{C}(x)[\partial]$. We say that L has 2-descent if $\exists f \in \mathbb{C}(x)$ with $\deg(f) = 2$ and $\exists \tilde{L} \in \mathbb{C}(f)[\partial_f]$ such that $L \sim_p \tilde{L}$.*

One could instead use the term "projective 2-descent" for this (because we use projective equivalence $\sim_p$) but we opted to use the shorter term.

Main goal: Let $L \in K[\partial]$ be irreducible. The goal of 2-descent is to give an explicit algorithm that can decide if L has 2-descent, and if so, find it (i.e. find $\tilde{L} \in \mathbb{C}(f)[\partial_f]$ with $L \sim_p \tilde{L}$ for some f of degree 2). Moreover, if L is defined over some field $C \subset \mathbb{C}$, we should only introduce algebraic extensions of C when necessary.

In the following sections, we limit L to be of order 2, unless otherwise specified. We will divide our algorithm into several steps. The first step is to find candidates for $\mathbb{C}(f)$ with $\deg(f) = 2$. Such a field is the fixed field of a Möbius transformation of order 2.

3.3 Möbius transformations

Proposition 1. *A Möbius transformation has order 2 if it is of the form $\sigma(x) = \frac{ax+b}{cx-a}$. Such σ has 2 fixed points in $\mathbb{C} \cup \{\infty\}$.*

One could apply a transformation that moves the fixed points of σ to $0, \infty$, which reduces σ to the notationally convenient $x \mapsto -x$. Our algorithm does not do this because it can introduce an unnecessary algebraic extension of the constants.

3.3.1 The singularity structure

Definition 18. *Let $L \in \mathcal{D}$ have order n. Assume p is a singularity of L. If there exists a basis of $V(L)$ of the form $e^{\int r} f_1, \ldots, e^{\int r} f_n$ where $r \in \mathbb{C}(x)$ and $f_1, \ldots, f_n$ are analytic at $x = p$, then p is called a* removable *singularity (also called* false *singularity). Otherwise p is called a* true *singularity.*

Suppose p is a singularity of L. If there exists a projectively equivalent $\tilde{L}$ for which p is a regular point, then p is a removable singularity. The true singularities of L are precisely those p that stay singular when L is replaced by any projectively equivalent operator.

Definition 19. *[23, 12] Let L be a second order differential operator, then for each singularity p of L, there are two generalized exponents e_1 and e_2. The exponent difference of L at $x = p$ is defined as $e_1 - e_2$. We denote it as $\Delta(L, p) = \pm(e_1 - e_2)$.*

Remark 16. *The $\pm$ sign appears because there is no canonical way to order the generalized exponents e_1 and e_2.*

Lemma 5. *Let L be a second order differential operator over $K = \mathbb{C}(x)$ and p be a singularity. The exponent difference $\Delta(L, p)$ modulo $\frac{1}{r}\mathbb{Z}$ is invariant under projective equivalence, where r is the ramification index.*

Proof. For a second order differential operator, the ramification index can be 1 or 2. When $r = 1$, then the generalized exponents are unramified, the proof can be found in Section 2.2 in [11]. When $r = 2$, the generalized exponents are ramified, then proof can be found in Section 3.3 in [30]. $\square$

Definition 20. *For any true singularity p, denote*

$$\text{type}(L, p) := \begin{cases} ''\text{irreg}'' & \textit{if} \quad \Delta(L, p) \notin \mathbb{C} \\ ''\text{irrat}'' & \textit{if} \quad \Delta(L, p) \in \mathbb{C} \setminus \mathbb{Q} \\ e \in [0, \frac{1}{2}] & \textit{if} \quad \Delta(L, p) \in \mathbb{Q} \end{cases}$$

Here, $e \in [0, \frac{1}{2}]$ such that $\Delta(L, p) \in (e + \mathbb{Z}) \cup (-e + \mathbb{Z})$.

Then we write the *singularity structure* of L as

$$S^{\text{type}} := \{(p, \text{type}(L, p)) \,|\, p \text{ true sing}\}.$$

Let π_i project on the i'th entry of S^{type}, then $S := \pi_1(S^{\text{type}}) \subseteq \mathbb{P}^1(\mathbb{C})$ denotes the set of true singularities of L.

Lemma 6. *If $L \sim_p \tilde{L} \in \mathcal{D}$ then L and $\tilde{L}$ have the same singularity structure S^{type} ([12, 28]).*

If $L \in C(x)[\partial]$ for some field $C \subset \mathbb{C}$, we denote:

$$M_{\mathbb{C}} := \{\sigma = \frac{ax+b}{cx-a} \,|\, a,b,c \in \mathbb{C} \text{ and } \sigma(S) = S\}$$

$$M_C := \{\sigma = \frac{ax+b}{cx-a} \,|\, a,b,c \in C \text{ and } \sigma(S) = S\}$$

$$M_{\mathbb{C}}^{\text{type}} := \{\sigma \in M_{\mathbb{C}} \,|\, \sigma(S^{\text{type}}) = S^{\text{type}}\}$$

$$M_C^{\text{type}} := \{\sigma \in M_C \,|\, \sigma(S^{\text{type}}) = S^{\text{type}}\}$$

places(C) $:=\{f \in C[x] \,| f$ is monic and irreducible$\}\bigcup\{\infty\}$.

Remark 17. places$(\mathbb{C}) \cong \mathbb{P}^1(\mathbb{C}) = \mathbb{C}\bigcup\{\infty\}$

If $\sigma \in \text{Aut}(C(x)/C)$ then σ acts on places(C) in a natural way, preserving degrees, which are defined as:

$$\deg(p) = \begin{cases} 1 & \text{if} \quad p = \infty; \\ \deg(p) & \text{if} \quad p \text{ is a polynomial} . \end{cases}$$

If $L = a_n\partial^n + \cdots + a_0\partial^0$ with $a_0, \ldots, a_n \in C[x]$, then computing the singularities as a subset of $\mathbb{P}^1(\overline{C}) \subset \mathbb{P}^1(\mathbb{C})$ would mean computing all roots (the splitting field) of a_n. The algorithm does not compute this splitting field because it could have exponentially high degree over C. Instead, it uses irreducible factors of a_n in $C[x]$ (and the point ∞) to represent the singularities, then we have the notation S_C^{type} and

$$M_C^{\text{type}} := \{\sigma \in M_C \,|\, \sigma(S_C^{\text{type}}) = S_C^{\text{type}}\}$$

To ensure that S is invariant under $\sim_p$ it is essential to discard all removable singularities.

Example 7. *Let $C = \mathbb{Q}$, and*

$$L := \partial^2 + \frac{12x^4+1}{x(2x^2-1)(2x^2+1)}\partial - \frac{8}{(2x^2-1)^2}$$

For this example we find

$$S^{\text{type}} := \{(\infty,0),(0,0),(\frac{-1}{\sqrt{2}},0),(\frac{1}{\sqrt{2}},0),(\frac{-1}{\sqrt{-2}},0),(\frac{1}{\sqrt{-2}},0)\}.$$

The set of true singularities is

$$S = \pi_1(S^{\text{type}}) = \{\infty, 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{-2}}, \frac{-1}{\sqrt{-2}}\}$$

Written in terms of places($\mathbb{Q}$) *it becomes*

$$S_C := \{\infty, x, x^2+\frac{1}{2}, x^2-\frac{1}{2}\} \subset \text{places}(\mathbb{Q}),$$

$$S_C^{\text{type}} := \{(\infty,0),(x,0),(x^2+\frac{1}{2},0),(x^2-\frac{1}{2},0)\}$$

and

$$M_C^{\text{type}} = \{-x, \frac{1}{2x}, \frac{-1}{2x}\}.$$

This example was quite easy because it has obvious 2-descent. Moreover, all singularities were true singularities with $\text{type}(L,p)=0$. Removable singularities are common in larger examples, such as Example 3 in Section 7. Using S instead of S_C would have introduced an extension of $C=\mathbb{Q}$ of degree 4 in this example, however, such an extension could have been much larger (e.g. if x^5-x-1 had appeared in the denominator of L, which has a splitting field of degree 120).

3.3.2 Finding candidates for σ

For $i=1,2,\ldots$, let S_i denote the set of all $p \in S_C$ with $\deg(p)=i$.

Algorithm: Compute Möbius transformations.

Input: The singularity structure S_C^{type}.

Output: The set M_C^{type}, i.e., the set of all $\sigma \in \text{Aut}(C(x)/C)$ of order 2 that fix S_C^{type}. (In this paper we omit 2-descent for σ's that are not defined over C because in that case it is better to compute a larger descent, of type $C_2\times C_2$, D_n, A_4, S_4, or A_5).

Step 1: Compute S_i from S_C^{type} and let n_i denote the number of elements of S_i.

Step 2: Let $n_{sing} := \sum i\, n_i$ (the total number of true singularities when counted in $\mathbb{P}^1(\overline{C})$).

Step 3: If $n_{sing} < 3$ then return "With < 3 singularities, descent is not necessary nor implemented" and stop.

Step 4: Now $n_{sing} \geq 3$.

1. If $n_1 \geq 3$, then call **Case1**
2. If $n_1 = 1, n_2 = 1$, then call **Case2**
3. If $n_1 = 2, n_2 = 1$, then call **Case3**
4. If $n_2 \geq 2$, then call **Case4**
5. If $n_i \geq 1$ for some $i \geq 3$, then call **Case5**

Algorithm: Case1.

Input: S_C^{type} with S_1 having ≥ 3 elements.

Output: The set M_C^{type}.

Before describing Algorithm Case1, first some remarks. In general $\sigma = \frac{ax+b}{cx+d}$ is determined by the image of three points $\sigma(p_1), \sigma(p_2), \sigma(p_3)$. Since we assume $|\sigma| = 2$, we can write $\sigma = \frac{ax+b}{cx-a}$. In general, such σ is determined by two points $\sigma(p_1)$, $\sigma(p_2)$ except in one case: when $\sigma(p_1) = p_2$, $\sigma(p_2) = p_1$. In that case one more point is needed to determine $\sigma = \frac{ax+b}{cx-a}$.

Algorithm Case1 will choose a pair $p_1, p_2 \in S_1$ ($p_1 \neq p_2$) and loops over all $n(n-1)$ pairs $q_1, q_2 \in S_1$ ($q_1 \neq q_2$). If the types of q_1, q_2 match those of p_1, p_2, the algorithm will compute the σ that maps p_1, p_2 to q_1, q_2. In the one case that $q_1, q_2 = p_2, p_1$, a third point p_3 is used to determine σ. There are $n-2$ choices for $\sigma(p_3)$, namely from $S_1 - \{p_1, p_2\}$. The number of computed σ's is then $\leq n(n-1) - 1 + (n-2)$ (equality if they all have the same type). Then we remove those σ for which S_C^{type} is not σ-invariant (That means remove all σ's that send a true singularity to a non-singular point or to a false singularity (Definition 18), and, remove all σ's that send a singularity to a singularity of a different type).

Algorithm: Case2

Input: S_C^{type} with S_1 having 1 element and S_2 having 1 element.

Output: The set M_C^{type}.

Step 1: Let the polynomial in S_2 be $x^2 + c_1 x + c_0$.

Step 2: Write $\sigma_1 = -\frac{c_1\,x+2c_0}{2x+c_1}$ and $\sigma_2 = \frac{a\,x+c_0\,c+c_1\,a}{c\,x-a}$.

> **Remark 18.** *σ_1 is the unique Möbius transformation of order 2 that fixes the roots of $x^2 + c_1 x + c_0$; σ_2 is the parameterized family of all σ of order 2 that swap the roots of $x^2 + c_1 x + c_0$.*

Step 3: Let p_1 be the one element of S_1. Equating $\sigma(p_1)$ to p_1 gives a linear equation that determines the values of the homogeneous parameters a, c in σ_2.

Step 4: Check which (if any) of σ_1, σ_2 fix S_C^{type} and return those.

Algorithm **Case3** is similar to Algorithm **Case2**.

Algorithm: Case4

Input: S_C^{type} with S_2 having ≥ 2 elements.

Output: The set M_C^{type}.

Step 1: Choose one polynomial from S_2. Denote it as $f_1 = x^2 + c_1\,x + c_0$.

Step 2: Do the following substeps $1-4$ to get the set T_1:

1. Write $\sigma_1 = -\frac{c_1\,x+2c_0}{2x+c_1}$ and $\sigma_2 = \frac{a\,x+c_0\,c+c_1\,a}{c\,x-a}$ (See the Remark in Algorithm Case2).
2. Choose another polynomial in S_2, and denote it as $f_2 = x^2 + d_1\,x + d_0$.
3. Write $\sigma_3 = -\frac{d_1\,x+2d_0}{2x+d_1}$ and $\sigma_4 = \frac{a\,x+d_0\,c+d_1\,a}{c\,x-a}$.
4. Let $a := d_0 - c_0$, $c := c_1 - d_1$, then $\sigma_2 = \sigma_4$ swaps the roots of f_1 as well as the roots of f_2.
   $T_1 := \{\sigma \in \{\sigma_1, \sigma_2, \sigma_3\} | \sigma \text{ fixes } S_C^{\text{type}}\}$.

Step 3: Denote the polynomials in S_2 as f_i, then $T_2 := \bigcup_{i=2}^{n_2} \text{FindMaps}(f_1, f_i)$

(See below for the subalgorithm **FindMaps**)

Step 4: $T_3 := \bigcup_{i=3}^{n_2} \text{FindMaps}(f_2, f_i)$.

Step 5: $T_1 \bigcup T_2 \bigcup T_3$.

Remark. Taking a set union means removing duplicates. The duplicates are the elements of T_3 that do not swap the roots of f_1, and σ_3 might also be duplicate (it could be in T_2 if $n_2 > 2$).

Subalgorithm: FindMaps

Input: Two irreducible polynomials $f, g \in C[x]$ of equal degree.

Output: All $\sigma \in M_C^{\text{type}}$ that map roots of f to roots of g.

1. Compute the roots of g in $C(\alpha) \cong C[x]/(f)$.
2. For each root β_j, compute $a, b, c \in C$ (not all 0) with $\frac{a\,\alpha+b}{c\,\alpha-a} = \beta_j$.
   This is done by computing coefficients (w.r.t α) of $a\,\alpha + b - \beta_j(c\,\alpha - a)$ and equating them to 0.
3. For each $\frac{a\,x+b}{c\,x-a}$ found in step 2 check if it fixes S_C^{type}, if so, include it in the output.

Algorithm: Case5

Input: S_C^{type} with S_i having ≥ 1 elements and $i \geq 3$.

Output: The set M_C^{type}.

Step 1: Find S_i for an $i \geq 3$ with $n_i > 0$.

Step 2: Choose a polynomial f in S_i. Denote $C(\alpha) \cong C[x]/(f)$, with $f(\alpha) = 0$.

Step 3: For each polynomial $g \in S_i$, call FindMaps(f, g). Then M_C^{type} would be $\bigcup_{g \in S_i}$ FindMaps(f, g).

3.4 Computing 2-descent, Case A

Definition 21. *Given two differential operators L_1, $L_2 \in C(x)[\partial]$, we define:*

$$\text{Hom}_{\mathcal{D}}(L_1, L_2) := \{\varphi : V(L_1) \to V(L_2) \,|\, \exists\; G \in \mathcal{D} \text{ s.t. } G(y) = \varphi(y) \text{ for all } y \in V(L_1)\}.$$

Notation 1. *We will usually use G to denote an element of* $\text{Hom}(L_1, L_2)$.

Remark 19. *Note that $G \in \text{Hom}(L_1, L_2)$ is equivalent to saying that $L_2 G$ is right divisible by L_1. Viewed this way, it is clear that if $G \in \text{Hom}(L_1, L_2)$ then $\sigma(G) \in \text{Hom}(\sigma(L_1), \sigma(L_2))$ for any automorphism σ.*

Remark 20. *Given φ, the operator G is not unique, because G and $G + L_1$ give the same map $V(L_1) \to V(L_2)$. However, if we require $\text{Ord}(G) < \text{Ord}(L_1)$ (which one can do using division with remainder), then φ uniquely determines G.*

Remark 21. *If φ in Definition 21 is a bijection between L_1 and L_2, then it is called a gauge transformation as defined in Chapter 2. In this case, the map $\varphi^{-1} : V(L_2) \to V(L_1)$ is also a gauge transformation, the corresponding operator $\tilde{G}$ with $\tilde{G}G : V(L_1) \to V(L_1)$ being the identity map can be found using the Extended Euclidean Algorithm [11].*

The above definition leads the following lemma for the irreducibility of the differential operator.

Lemma 7. *Given a differential operator $L \in C(x)[\partial]$. If L is irreducible in $\overline{C}(x)[\partial]$, then* $\dim(\mathrm{Hom}_{\overline{C}(x)[\partial]}(L, L)) = 1$.

Notations: In the rest of this chapter, let $L \in C(x)[\partial]$ have order 2, and be irreducible (even in $\mathbb{C}(x)[\partial]$). Let $\sigma \in \mathrm{Aut}(C(x)/C)$ have order 2 and fixed field $C(f) \subset C(x)$.

Lemma 8. *If $\exists \tilde{L} \in \mathbb{C}(f)[\partial_f]$ with $L \sim_p \tilde{L}$, then $L \sim_p \sigma(L)$.*

Proof. $L \sim_p \tilde{L} = \sigma(\tilde{L}) \sim_p \sigma(L)$. □

So if not $L \sim_p \sigma(L)$ then $L \in C(x)[\partial] \subset \mathbb{C}(x)[\partial]$ does not descend to $\mathbb{C}(f)$. If $L \sim_p \sigma(L)$ then we will consider two cases:

Notation 2. ***Case A*** *is when there exists $G = r_0 + r_1\,\partial \in \mathbb{C}(x)[\partial]$ such that $G(V(L)) = V(\sigma(L))$, i.e. $L \sim_g \sigma(L)$.*

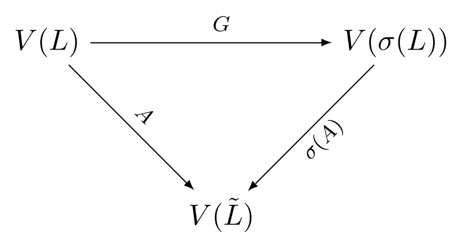
Case B *is when there exists $G = e^{\int r} \cdot\ (r_0 + r_1\,\partial)$ such that $G(V(L)) = V(\sigma(L))$, i.e. $L \sim_p \sigma(L)$.*

*(**Note:** Case A $\Rightarrow$ Case B.)*

This section treats only Case A. Section 3.5 will reduce Case B to Case A.

In **Case A**, when $L \sim_g \sigma(L)$, it is known in [21] that there exists $\tilde{L} \in \mathbb{C}(f)[\partial_f]$ with $\tilde{L} \sim_g L$. Then we have the following diagram:

Diagram 1

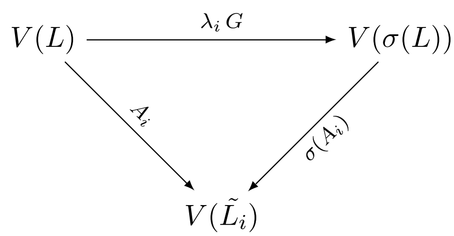


Here, A, $\sigma(A)$, and $\tilde{L}$ are unknown. Whether or not such a diagram commutes is studied in Theorem 4 below.

Remark 22. *A gauge transformation is a bijective map $A : V(L) \to V(\tilde{L})$ that can be represented by a differential operator in $\mathbb{C}(x)[\partial]$. So we can define $\sigma(A)$ simply by applying σ to the operator that represents the map A.*

Theorem 4. *Let L and σ be as before, and $G : V(L) \to V(\sigma(L))$ be a gauge transformation. Suppose $\tilde{L}_1$, $\tilde{L}_2 \in \mathbb{C}(f)[\partial_f]$ and $A_i : V(L) \to V(\tilde{L}_i)$ are gauge transformations. Then:*

1. *For each $i = 1, 2$, there is exactly one $\lambda_i \in \mathbb{C}^*$ such that the following diagram commutes.*
   Diagram 2



2. *If $\tilde{L}_1 \sim_g \tilde{L}_2$ over $\mathbb{C}(f)$, then $\lambda_1 = \lambda_2$; Otherwise, $\lambda_1 = -\lambda_2$.*
3. *In particular, $\{\lambda_1, -\lambda_1\}$ depends only on (L, σ, G).*

Proof. First consider the diagram without λ_i in it. In it we find two gauge transformations $V(L) \to V(\tilde{L}_i)$, namely A_i and $\sigma(A_i)G$. After choosing bases of $V(L)$ and $V(\tilde{L}_i)$, we can view these gauge transformations as bijections: $\mathbb{C}^2 \to \mathbb{C}^2$. Then by linear algebra, there is a constant $\lambda_i \in \mathbb{C}^*$ such that the map:

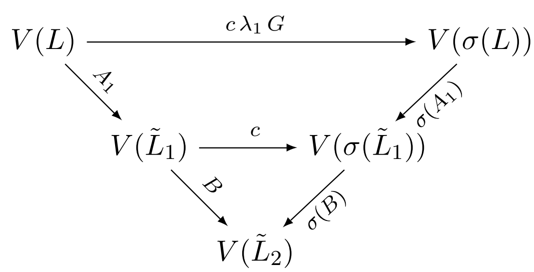
$$A_i - \lambda_i \sigma(A_i)G : V(L) \to V(\tilde{L}_i). \tag{3.3}$$

has a non-zero kernel. The kernel of (3.3) corresponds to a right hand factor of L, namely, the GCRD of L and the operator in (3.3). However, L is irreducible so this kernel must be $V(L)$ itself. That means Diagram 2 commutes. That λ_i is unique follows from linear algebra: there can be at most one λ_i for which (3.3) is the zero map. Item 1 follows.

For item 2, since $\tilde{L}_1 \sim_g L \sim_g \tilde{L}_2$, there exists a gauge transformation $B : V(\tilde{L}_1) \to V(\tilde{L}_2)$. This B is unique up to multiplying by a constant that we choose in such a way that the composition $BA_1 : V(L) \to V(\tilde{L}_2)$ coincides with A_2. Since $\sigma(\tilde{L}_1) = \tilde{L}_1$, $\sigma(\tilde{L}_2) = \tilde{L}_2$ one sees that $\sigma(B)$ maps $V(\tilde{L}_1)$ to $V(\tilde{L}_2)$ as well. So $\sigma(B)$ must be $c \cdot B$ for some $c \in \mathbb{C}^*$. Then $|\sigma| = 2$ implies that $c = \pm 1$. Now $c = 1$ iff $\sigma(B) = B$ iff $B \in \mathbb{C}(f)[\partial_f]$ iff $\tilde{L}_1, \tilde{L}_2$ are gauge-equivalent over $\mathbb{C}(f)$. Otherwise, if $c = -1$, then $B \notin \mathbb{C}(f)[\partial_f]$ and $\tilde{L}_1, \tilde{L}_2$ are gauge-equivalent over $\mathbb{C}(x)$ but not over $\mathbb{C}(f)$. To prove item 2 we now have to show that $\lambda_2 = c\lambda_1$.

If λ_i is such that Diagram 2 commutes (for $i = 1, 2$) then the following diagram commutes:

Diagram 3



The composed map $B\,A_1$ at the left of Diagram 3 coincides with the map A_2 in Diagram 2 for $i = 2$. Applying σ to $B\,A_1$ and A_2, we see that the composed map at the right of Diagram 3 coincides with the map $\sigma(A_2)$ in Diagram 2 for $i = 2$. Then the maps at the top of Diagram 3 and Diagram 2 for $i = 2$ must coincide as well, i.e., $\lambda_2 G = c\lambda_1 G$. Hence $\lambda_2 = c\lambda_1$. Item 2 (and hence item 3) follow. $\square$

3.4.1 Algorithm for finding 2-descent in Case A

Notations L, C, G, σ, A are as in above. Our goal is to compute 2-descent: $L \sim_p \tilde{L} \in \mathbb{C}(f)[\partial_f]$. Here f is determined from σ as in Remark 14. We will compute $A : V(L) \to V(\tilde{L})$ first, then use A to find $\tilde{L}$.

Algorithm: Case A for computing a 2-descent $\tilde{L}$ for L.

Input: L, G, σ and C.

Output: $\tilde{L}$ and A, defined over an optimal extension of C.

Step 1: Write $A = (a_{00} + a_{01}x)\partial + (a_{10} + a_{11}x)$, with a_{00}, a_{01}, a_{10}, a_{11} unknowns (which will take values in $\mathbb{C}(f)$).

Step 2: The operator $A - \lambda\sigma(A)G$ in (3.3) should vanish on $V(L)$, so the remainder of $A - \sigma(A)\lambda G$ right divided by L must be 0. This remainder is of the form $(R_{00} + R_{01}x)\partial^0 + (R_{10} + R_{11}x)\partial$, where the R_{ij} are $C(\lambda, f)$-linear combinations of a_{ij}. This produces a system of 4 equations $R_{ij} = 0$ in 4 unknowns a_{ij}.

Step 3: To have a nontrivial solution, the corresponding 4×4 matrix M must have determinant 0. Equating $\det(M)$ to 0 gives a degree 4 equation for λ. Solve for λ.
Remark. The equation for λ is of the form $(\lambda^2 - a)^2 = 0$, where $a = \lambda_1^2 = \lambda_2^2$ with λ_1, λ_2 as in Theorem 4. If L and σ are defined over a field $C \subseteq \mathbb{C}$ then $\tilde{L}$ and A are defined over $C(\sqrt{a})$.
If $\sqrt{a} \notin C$ then it follows from Theorem 4 that the extension by $\lambda_i = \pm\sqrt{a}$ is necessary.

Step 4: Plug in one value for λ in M, then solve M to find values for $a_{00}, a_{01}, a_{10}, a_{11}$ in $C(\sqrt{a}, f)$.

Step 5: Compute $\mathrm{LCLM}(A, L)$ to obtain $\tilde{L}A$. Right divide by A to find $\tilde{L} \in C(\sqrt{a}, f)[\partial_f]$.

Step 6: (optional) Introduce a new variable, say x_1, and compute an operator $L_{x_1} \in C(\sqrt{a}, x_1)[\partial_{x_1}]$ that corresponds to $\tilde{L}$ under the change of variables $x_1 \mapsto f$.

3.5 Computing 2-descent, Case B

Definition 22. *Let L_1, $L_2 \in \mathcal{D} = K[\partial]$. The* symmetric product *$L_1 \circledS L_2$ is defined as the monic differential operator in $\mathcal{D}$ with minimal order for which $y_1 y_2 \in V(L_1 \circledS L_2)$ for all $y_1 \in V(L_1)$, $y_2 \in V(L_2)$.*

Lemma 9. *If $L = \partial^2 + c_0 \in C(x)[\partial]$, and $G := e^{\int r} \cdot (r_0 + r_1\partial)$ is a bijection from $V(L)$ to $V(\sigma(L))$, then $(e^{\int r})^2$ is a rational function.*
If $L := \partial^2 + a_1\partial + a_0 \in \mathbb{C}(x)[\partial]$, then $L_1 := L \circledS (\partial - \frac{1}{2}a_1)$ is of the form $\partial^2 + c_0$ (with $c_0 = a_0 - \frac{1}{4}a_1^2 - \frac{1}{2}a_1')$.

The proof of the lemma follows by computing the effect of G on the Wronskian, and the fact that the Wronskians of $\partial^2 + c_0$ and $\sigma(\partial^2 + c_0)$ are rational functions (1 and $\sigma(x)'$ respectively).

Let $L \in C(x)[\partial]$ irreducible (even over $\mathbb{C}$) and of order 2, and $\sigma \in \mathrm{Aut}(C(x)/C)$ of order 2. The implementation equiv [18] can check if $L \sim_p \sigma(L)$, and if so, find $r, r_0, r_1 \in C(x)$ for which $G := e^{\int r} \cdot (r_0 + r_1\partial)$ is a bijection from $V(L)$ to $V(\sigma(L))$. Assume that such σ

and G are given. After the simple transformation in the lemma above, we may assume that $(e^{\int r})^2$ is a rational function.

If $e^{\int r}$ itself is a rational function, then we are in Case A. Otherwise, we can write $e^{\int r} = p(x)\sqrt{f(x)}$ for some square-free polynomial $f(x)$, and some $p(x) \in C(x)$.

Definition 23. *The branch points of G are the roots of $f(x)$, and ∞ if $f(x)$ has odd degree.*

To reduce Case B to Case A, we have to eliminate the branch points. Our algorithm will first eliminate all branch points that can be eliminated without a field extension of C. It will only extend C if there is no descent w.r.t. σ defined over C.

3.5.1 Branch points

It is convenient to view the set of branch points as a subset of $\mathbb{P}^1(\overline{C})$. However, to avoid splitting fields, the algorithm represents the branch points with a set $B \subset \text{places}(C)$ instead. This B is the set of irreducible factors of $f(x)$ in $C[x]$, as well as ∞ if $f(x)$ has odd degree. The goal is to eliminate branch points until we reach $B = \emptyset$, i.e., Case A.

Definition 24. *If $\sigma(\infty) = \infty$, then denote* $\text{Inf} := \{\infty\}$, *otherwise* $\text{Inf} := \{\infty, x - \sigma(\infty)\}$. *Denote $B_I = B \bigcap \text{Inf}$ and $B_N = B \setminus B_I$.*
Let $f_1(x), f_2(x) \in B_N$. We say that $f_1(x)$ matches $f_2(x)$ when the roots of $f_2(x)$ are the same as the roots of $f_1(\sigma(x))$ (i.e. the numerator of $f_1(\sigma(x))$ is f_2).
If $\sigma(\infty) \neq \infty$, then we say that the polynomial $x - \sigma(\infty)$ matches ∞.

Lemma 10. *If $f_1(x) \neq f_2(x) \in B_N$ and $f_1(x)$ matches $f_2(x)$, then B_N turns into $B_N \setminus \{f_1, f_2\}$ when we replace L by $L_{\text{new}} := L \circledS (\partial - \frac{1}{2} \cdot \frac{f_1(x)'}{f_1(x)})$.*

Proof. The composed transformation

$$V(L_{\text{new}}) \to V(L) \to V(\sigma(L)) \to V(\sigma(L_{\text{new}}))$$

is

$$\sqrt{\sigma(f_1)} \cdot G \cdot \frac{1}{\sqrt{f_1}}.$$

The polynomial f equals $f_1 f_2 \cdots$ where the $\cdots$ refer to the other factors of f in $B \setminus \{\infty\}$. The transformation G is of the form $\sqrt{f_1 f_2 \cdots} \cdot (r_0 + r_1 \partial)$. Factors can be removed from

the square-root in G either by division or by multiplication by a square-root (factors in $C(x)$ can be moved to r_0, r_1). So in the composed transformation, the factors f_1 and f_2 will disappear from the square-root in G (note: this uses the assumption $f_1 \neq f_2$ (which implies that their gcd is 1 since they are monic irreducible polynomials)).
A subtlety is that if $\sigma(\infty) \neq \infty$, then $\sigma(f_1)$ is not f_2 but $cf_2/(x-\sigma(\infty))^d$, for some $c \in C$, where d is the degree of f_1 and f_2. This means that if $\sigma(\infty) \neq \infty$ and d is odd, then the set B_I will change when we replace L by L_{new} ($B_I = \emptyset$ will change to Inf, and $B_I = \text{Inf}$ will change to $\emptyset$). □

Lemma 11. *If $\sigma(\infty) \neq \infty$, and $B_I = \{\infty, f_1\}$ (here $f_1 = x - \sigma(\infty)$) then the factor f_1 inside the square root in G will cancel out (i.e. B_I will become $\emptyset$) if we replace L by $L_{\text{new}} := L \circledS (\partial - \frac{1}{4} \cdot \frac{1}{f_1})$.*

Proof. The solutions of L_{new} differ a factor $\sqrt[4]{f_1}$ from the solutions of L. The lemma follows from a similar computation as the proof of Lemma 10, except that this time $\sigma(f_1)$ is of the form c/f_1 for some constant c. Thus, the composed map is of the form $\sqrt[4]{c/f_1} \cdot G \cdot 1/\sqrt[4]{f_1}$, and $\sqrt{f_1}$ is canceled from the square root in G. □

In the following algorithm, L and σ are as in Section 4, and $G = e^{\int r} \cdot (r_0 + r_1 \partial)$ with $r, r_0, r_1 \in C(x)$.

Algorithm: Case B for computing a 2-descent $\tilde{L}$ for L.

Input: L, G, σ and C.

Output: $\tilde{L}$ and A (defined over C whenever possible).

Step 1 Initialization: If $(e^{\int r})^2$ is not a rational function, then replace L by $L \circledS (\partial - \frac{1}{2} \cdot \frac{a_1}{a_2})$ as in Lemma 9 and update G accordingly.
Rewrite G as $\sqrt{f(x)}(r_0 + r_1 \partial)$ with $f(x)$ monic and square-free (updating $r_0, r_1 \in C(x)$ to move any rational factor from $e^{\int r}$ to r_0, r_1).
If $f(x) = 1$ then call **Case A** and stop.

Step 2: Factor $f(x)$ in $C[x]$ to find $B, B_I, B_N \subset \text{places}(C)$.

Step 3: $g :=$**Findg**(B_N, σ, C).
(See below for the subalgorithm **Findg**)

Step 4: Let $h := \frac{1}{2} \cdot \frac{g\prime}{g}$. Replace L by $L Ⓢ (\partial - h)$ and update G, B, B_I, B_N accordingly. Now B_N should be $\emptyset$.

Step 5: If $B_I \neq \emptyset$ then let $h := \frac{1}{4} \cdot \frac{1}{f_1}$ with f_1 as in Lemma 11. Replace L by $L Ⓢ (\partial - h)$ and update G, B accordingly. Now B should be $\emptyset$.

Step 6: Call **Case A**.

Subalgorithm: Findg.

Input: B_N, σ, C.

Output: g.

Step 1: If $B_N = \emptyset$, return 1 and stop.

Step 2: Else, for each $P_i \in B_N$,

1. Find its matched (Def. 24) element $P_j \in B_N$.
2. If $P_i \neq P_j$ then $g :=$ **Findg**$(B_N \setminus \{P_i, P_j\}, \sigma, C)$, return $g \cdot P_i$ and stop.

Step 3: Now each $P \in B_N$ matches itself, and hence has even degree. Choose $P \in B_N$ with minimal degree, and let $\alpha \in \overline{C}$ be one root of P, so $C(\alpha) \cong C[x]/(P)$. Let B_N^α be the set of all irreducible factors in $C(\alpha)[x]$ of all elements of B_N. Return **Findg**$(B_N^\alpha, \sigma, C(\alpha))$.

3.6 Main Algorithm of 2-descent

Given a second order irreducible linear differential operator $L \in \mathbb{C}(x)[\partial]$, we can now decide if there exists a 2-descent for L, and if so, we can find this descent.

Algorithm 2-descent.

Input: A second order irreducible differential operator $L \in C(x)[\partial]$ and the field C.

Output: descent, if it exists for some $\sigma \in \mathrm{Aut}(C(x)/C)$ of order 2.

Step 1: Compute the set of true singularities, and the singularity structure S_C^{type}.

Step 2: Call **Compute Möbius transformations** in Section 3.3 to compute the set M_C^{type}.

Step 3: For each $\sigma \in M_C^{\mathrm{type}}$, call [18] to check if $L \sim_p \sigma(L)$, and if so, to find $G : V(L) \to V(\sigma(L))$.
If we find σ with $L \sim_p \sigma(L)$, then call algorithm Case B in Section 3.5 and stop.

3.7 Examples of 2-descent

We give two examples. The first example is easy (it has $G = r_0 + r_1\partial$ with $r_1 = 0$). The second one is less trivial. The first example is in **Case A** as in Section 3.4, the second example involves both **Case A** and **Case B**.

Example 8. *Let*

$$L = \partial^2 + \frac{28x-5}{x(4x-1)}\partial + \frac{144x^2+20x-3}{x^2(4x-1)(4x+1)}$$

Step 1: *Compute the singularity structure of L*

$$S_C^{\text{type}} := \{(x,0),(\infty,0),(x-\frac{1}{4},0),(x+\frac{1}{4},0)\}$$

Step 2: *Compute Möbius transformations. Since S_1 has $n_1 = 4$ elements, we end up in algorithm* ***Case1*** *of Section 3.3 which produces:*

$$\{-x, \frac{-1}{16x}, \frac{1}{16x}, \frac{-1}{4}\frac{4x-1}{4x+1}, \frac{1}{4}\frac{4x+1}{4x-1}\}$$

Step 3: *There are 5 choices for σ. The first one is $x \mapsto -x$ corresponding to the subfield $C(f) = C(x^2)$. The* ***equiv*** *[18] program finds $G = \frac{4x-1}{4x+1}$. Next we compute $A := -4x^2 + x$, and then $\tilde{L}$. After applying a change of variable $x \mapsto \sqrt{x_1}$ the result reads*

$$L_{x_1} := (16x_1 - 1)x_1\partial^2 + (32x_1 - 2)\partial + 4$$

which has 3 true singularities and is easy to solve.

Example 9. *Consider the operator:*

$$L := \partial^2 + \frac{4(1296x^5+576x^4-144x^3-72x^2+x+1)}{x(6x-1)(2x+1)(6x+1)(12x^2-1)}\partial + \frac{2(5184x^6-864x^5-1656x^4+48x^3+162x^2+6x-1)}{(-1+2x)x^2(6x-1)(2x+1)(6x+1)(12x^2-1)}$$

Step 1: *Compute the singularity structure of L*

$$S_C^{type} := \{(x,0),(\infty,0),(x-\frac{1}{2},0),(x+\frac{1}{2},0),(x-\frac{1}{6},0),(x+\frac{1}{6},0)\}$$

($12x^2 - 1$ is a removable singularity, Definition 18).

Step 2: *Compute Möbius transformations. Since* S_1 *has* $n_1 = 6$ *elements, we are again in Case1, and find:*

$$\{-x, \frac{-1}{12x}, \frac{1}{12x}, \frac{-1}{2}\frac{2x-1}{6x+1}, \frac{1}{2}\frac{2x+1}{6x-1}, \frac{-1}{6}\frac{6x-1}{2x+1}, \frac{1}{6}\frac{6x+1}{2x-1}\}$$

Step 3: *The first* σ *we try is* $x \mapsto -x$. *The* ***equiv*** *program finds*

$$G := \frac{x(12x^2+4x-1)}{12x^2-1}\partial + \frac{3}{2}\frac{(2x+1)(10x-1)}{12x^2-1}$$

so $G(V(L)) = V(\sigma(L))$. *Then compute a 4 by 4 matrix from the linear equations for the* a_{ij}, *equate the determinant to 0 and find* $\lambda = \pm 2$. *We choose* $\lambda = 2$ *and find*

$$A := (-36x^4 - \frac{1}{4} + 10x^2)\partial + 1 - \frac{1}{4}\frac{(288x^4 + 1 - 84x^2)}{x}.$$

We get

$$\begin{aligned} L_{x_1} :=& 4x_1^2(-1+36x_1)(4x_1-1)(12x_1-1)^2\partial^2 + \\ & 8x_1(12x_1-1)(4x_1-1)(216x_1^2-54x_1+1)\partial - \\ & 3 - 2544x_1^2 + 10368x_1^3 + 48x_1 \end{aligned}$$

which is $\tilde{L} \in C(x^2)[\partial_{x^2}]$ *rewritten with* $x \mapsto \sqrt{x_1}$. *This* L_{x_1} *has 4 true singularities, and allows a further 2-descent. Applying steps (1)(2)(3) to* L_{x_1} *again, we are actually in* ***Case B*** *as in Section 3.5, applying the algorithm (details are given in a Maple worksheet [13]) we find a new operator* $\tilde{L_1} \sim_p L_{x_1}$ *defined over the subfield* $\mathbb{C}(f_1)$ *where* $f_1 := x_1 + \frac{1}{144x_1}$. *Replacing* f_1 *by a new variable* x_2 *we get:*

$$\begin{aligned} L_{x_2} :=& 4(36x_2+11)(18x_2-5)(6x_2+1)(6x_2-1)^2\partial^2 + \\ & 36(6x_2-1)(1296x_2^3+1620x_2^2+20x_2-9)\partial + \\ & 34992x_2^3 - 207036x_2^2 - 2331 + 3456x_2 \end{aligned}$$

which has 3 true regular singularities (as well as a few removable singularities).

CHAPTER 4

AN IMPROVED ALGORITHM FOR COMPUTING 2-DESCENT, CASE A

In Chapter 3, we use algorithm *Case A* to compute the 2-descent when there is only gauge equivalence involved between our input operator L and $\sigma(L)$. This algorithm computes a set of linear equations to determine $A = (a_{10} + a_{11}x)\partial + (a_{00} + a_{01}x)$ and a constant λ. Once we have A we get $\tilde{L}$ correspondingly. In Step 4 of algorithm *Case A*, we select one $(a_{00}, a_{01}, a_{10}, a_{11})$ from a vector space of dimension 2. So our output $\tilde{L}$ is just one member of a 2-dimensional set of possible outcomes. The question for this situation is: Which A gives us the shortest $\tilde{L}$? Without a good answer, we can not expect the output to be of optimal size. The improved algorithm proposed in this chapter will avoid computing linear equations and usually produce a smaller output $\tilde{L}$.

4.1 An improved algorithm for Case A

In this section, we still assume that $L \in C(x)[\partial]$ has order 2, and is irreducible. Let $\sigma \in \mathrm{Aut}(C(x)/C)$ have order 2 and fixed field $C(f) \subset C(x)$.

The goal of 2-descent problem is: find $\tilde{L} \in \overline{C}(f)[\partial_f]$ that is projectively equivalent to L. The 2-descent $\tilde{L}$ is not unique; any operator equivalent to $\tilde{L} \in \overline{C}(f)[\partial_f]$ is also a solution of the 2-descent problem. In our improved algorithm we add additional conditions which will limit the number of output candidates. In experiments, this usually turns out to lead shorter output.

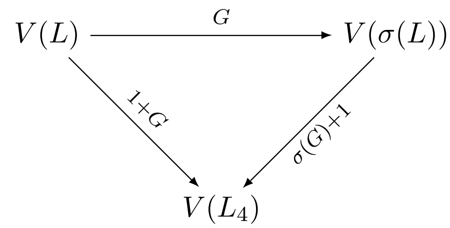
Suppose $G : V(L) \to V(\sigma(L))$ is a gauge transformation, where σ has order 2. We have the following fact.

Lemma 12. *Given a second order differential operator L. If $V(L) = V(\sigma(L))$ then $\exists c \in C(x) - \{0\}$ such that $cL = \sigma(cL)$.*

Proof. Take $c = ((f')^2 \cdot \text{leading coefficient}(L, \partial))^{-1}$. Then cL and $\sigma(cL)$ have the same leading coefficient. Since $V(cL) = V(L) = V(\sigma(L)) = V(\sigma(cL))$, then by Lemma 3, we have $cL = \sigma(cL)$. □

Let $L_4 := \text{LCLM}(L, \sigma(L)) \in C(f)[\partial_f]$ then $V(L_4) = V(L) + V(\sigma(L))$. The order of L_4 is 4 except if $V(L) = V(\sigma(L))$, but we will exclude that case since descent is trivial in that case (see Lemma 12). Consider the following diagram:

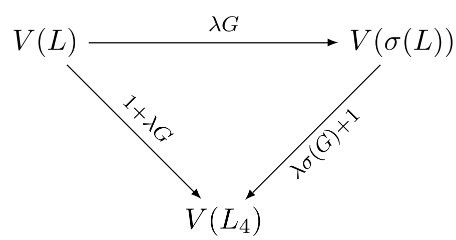
Diagram 4



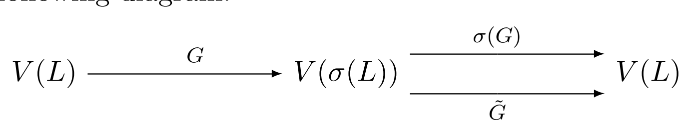
In the above Diagram 4, we have $G \in \text{Hom}(L, \sigma(L))$, so $1 + G \in \text{Hom}(L, L_4)$. Similarly, after applying σ to $1 + G$ (see also Remark 19), we know $1 + \sigma(G) \in \text{Hom}(\sigma(L), L_4)$. The question is still Diagram 4 commutes.

Lemma 13. *Given a second order irreducible differential operator L and second order automorphism σ as in Section 3.4, and a gauge transformation $G : V(L) \to V(\sigma(L))$, then there exist a constant λ such that the following diagram commutes.*

Diagram 5



Proof. Let $\tilde{G} : V(\sigma(L)) \to V(L)$ be the inverse of $G : V(L) \to V(\sigma(L))$ as in Remark 21. Since $G \in \text{Hom}(L, \sigma(L))$ and σ has order 2, then $\sigma(G) \in \text{Hom}(\sigma(L), L)$ by Remark 19. This time we have the following diagram:



From this diagram, we have two gauge transformations of $V(L)$ to itself, namely 1 and $\sigma(G)G$. After choosing a basis of $V(L)$, we can view these two gauge transformations as bijections of $\mathbb{C}^2$. Then by linear algebra, there is a constant b such that the map:

$$1 - b\sigma(G)G : V(L) \to V(L). \tag{4.1}$$

has a non-zero kernel. The kernel of (4.1) corresponds to a right hand factor of L, namely, the GCRD of L and the operator in (4.1). However, L is irreducible so this kernel must be $V(L)$ itself. That means there is a unique b such that (4.1) is the zero map, i.e. $b\sigma(G)G$ is an identity map. Let's take $\lambda = \sqrt{b}$ (or $-\sqrt{b}$). Now consider the map $0 = 1 - \lambda^2\sigma(G)G=$ $1 + \lambda G - ((1 + \lambda\sigma(G))\lambda G)$ from $V(L)$ to $V(L_4)$. This shows that Diagram 5 commutes. □

Theorem 5. *Given a second order differential operator L with σ, G, λ as in Lemma 13. Then there exists a second order differential operator $\tilde{L}$ such that $\tilde{L}$ is invariant under σ and $(1 + \lambda G)V(L) = V(\tilde{L})$.*

Proof. Let M :=LCLM$(L, 1 + \lambda G)$. Then there exists a second order differential operator $\tilde{L}$ such that $M = \tilde{L}(1 + \lambda G)$. Then $1 + \lambda G$ sends $V(L)$ to $V(\tilde{L})$, that means $V(\tilde{L}) \subseteq V(L_4)$. Next we will prove that $\tilde{L}$ invariant under σ. Since Diagram 5 is commutative, therefore $(1 + \sigma(\lambda G))V(\sigma(L))$=$V(\sigma(\tilde{L}))$ is the same as $(1 + \lambda G)(V(L))$=$V(\tilde{L})$. i.e $V(\sigma(\tilde{L})) = V(\tilde{L})$. Then we can conclude that after a suitable scaling $\tilde{L}$ is invariant under σ by Lemma 12. □

From the proof of Lemma 13 and Theorem 5, we obtain the following algorithm:

Algorithm: Improved Case A.

Input: L, G, σ and C.

Output: $\tilde{L}$, defined over an optimal extension of C.

Step 1: b' :=the remainder of $\sigma(G)G$ right divided by L (b' is a constant, it corresponds to $\frac{1}{b}$ from Lemma 13).

Step 2: Take $\lambda = \frac{1}{\sqrt{b'}}$ ($-\frac{1}{\sqrt{b'}}$ is another option).

Step 3: Compute LCLM$(1 + \lambda G, L)$ to obtain $\tilde{L}(1 + \lambda G)$. Right divide by $1 + \lambda G$ to find $\tilde{L} \in C(\sqrt{b'}, f)[\partial_f]$.

Step 4: (optional) Introduce a new variable, say x_1, and compute an operator $L_{x_1} \in C(\sqrt{b}, x_1)[\partial_{x_1}]$ that corresponds to $\tilde{L}$ under the change of variables $x_1 \mapsto f$.

4.2 Application for Second Order Differential Operators

As we stated at the beginning of this chapter, we assume the second order differential equation L is in Case A after we found the Möbius transformation σ of order 2 and the gauge transformation G between L and $\sigma(L)$. For Case B, we still adopt the algorithm as stated in Section 3.5 to reduce Case B to Case A.

For the second order differential operators that can be descent by the algorithm from Chapter 3 "Case A for computing a 2-descent $\tilde{L}$ for L" can also be handled by this new improved algorithm. This section will compare these two algorithms by using the example in the following page.

Example 10. *Consider the operator*

$$
\begin{aligned}
L := {} & (293760x^7 - 131976x^6 + 52704x^5 - 768x^4 - 5934x^3 + 1536x^2 - 141x + 4)(24x^2 - 1) \\
& (24x^2 + 1)(15x^2 - 12x + 2)^2 x^2 (x - 1)(3x - 1)(5x - 1)\partial^2 + \\
& 2x(15x^2 - 12x + 2)(228427776000x^{16} - 570570220800x^{15} + 602649020160x^{14} - \\
& 369855875520x^{13} + 139073599296x^{12} - 21164284260x^{11} - 8253615204 * x^{10} + \\
& 6008280732x^9 - 1783030374x^8 + 306396972x^7 - 25463352x^6 - 2334069x^5 + 1189734x^4 - \\
& 202401x^3 + 19026x^2 - 960x + 20)\partial + \\
& 6(2855347200000x^{18} - 8603249760000x^{17} + 11530367884800x^{16} - 9209871460800x^{15} + \\
& 4758150957696x^{14} - 1429804773504x^{13} + 33955024284x^{12} + 196108583976x^{11} - \\
& 104762250864x^{10} + 31219995024x^9 - 6220268763x^8 + 851673432x^7 - 71711194x^6 + \\
& 887473x^5 + 722720x^4 - 110630x^3 + 8460x^2 - 368x + 8)
\end{aligned}
$$

which is in $C(x)[\partial]$, where $C = \mathbb{Q}$.

Compute the singularity structure of L

$$S_C^{\text{type}} := \{(x, 0), (x^2 - \frac{1}{24}, 0), (x^2 + \frac{1}{24}, 0)\}$$

Compute Möbius transformations σ***.*** *Since* S_2 *has* $n_2 = 2$ *elements, we end up in algorithm* ***Case4*** *of Section 3.2 which produces:*

$$\{-x\}$$

Next, we compute gauge transformation between L *and* $\sigma(L)$*, we have*

$$G := 2(15x^2 - 12x + 2)(183600x^6 - 16074x^4 + 15x^2 + 14)x^2(24x^2 + 1)(5x - 1)(3x - 1)$$
$$(x - 1)\partial + 10905840000x^{14} - 22058811000x^{13} + 15983811000x^{12} - 5250125160x^{11} + 812$$
$$324646x^{10} - 36081666x^9 - 47254221x^8 + 34684731x^7 - 10450293x^6 + 1048791x^5 + 17298$$
$$0x^4 - 53432x^3 + 3484x^2 + 156x - 16$$

We use Algorithm "Case A" to produce $\tilde{L}$ *as follows which is sitting in the subring* $\mathbb{Q}(x_1)[\partial_{x_1}]$ *(where* x_1 *represents* x^2*):*

$$\tilde{L}_1 := x_1^2(24x_1 - 1)(24x_1 + 1)(-10626876000x_1^7 + 63527793750x_1^8 + 52 + 188x_1 - 1887743124$$
$$x_1^5 + 802266795x_1^6 - 144191x_1^2 + 123638094x_1^4 + 1154836x_1^3)(337863885284278080000x_1^{12} +$$
$$1116449220606847488 0x_1^{11} - 580795413631130880x_1^{10} - 144483644561742720x_1^9 +$$
$$1081610867047824x_1^8 + 909455229664560x_1^7 - 49295733535944x_1^6 + 484813472760x_1^5 +$$
$$28547793075x_1^4 - 764703410x_1^3 + 1536111x_1^2 + 119780x_1 - 836)(225x_1^2 - 84x_1 + 4)^2\partial^2 +$$
$$2x_1(225x_1^2 - 84x_1 + 4(347776 - 41223648x_1 + 973595573895266289349068960000000000$$
$$x_1^{24} + 3025040164853704295770523306880000 00x_1^{21} - 2101827610412775302008376664 0$$
$$0000000x_1^{23} + 142348060802989817114604150260736 0x_1^{19} - 228952142455974767906 48$$
$$4355695360x_1^{18} - 2253512624x_1^2 - 1964395078919206x_1^5 + 69067782015691945x_1^6 + 280$$
$$6272132256900113x_1^7 - 2239682169757352057 10x_1^8 + 2273602097570737769532x_1^9 + 19$$
$$68507281037623967823 95x_1^{10} - 68925344748975357874397 01x_1^{11} + 27899632327308274$$
$$441196406x_1^{12} + 27438808665877029805166098 38x_1^{13} - 593426116804027115320067230$$
$$92x_1^{14} + 31923974142242572653612931644 0x_1^{15} + 2862503876942953519772991317904x_1^{16}$$
$$- 152398285440x_1^4 + 809629852713545176680181262860800x_1^{20} - 206885659267433441 88$$
$$855508948560x_1^{17} - 1481436198722829353869962269280000 00x_1^{22} + 423419352832x_1^3)\partial -$$

$$
\begin{aligned}
&4997077687548768180765544821360000000x_1^{24} - 48694225859441867217408354291447 3\\
&600x_1^{21} + 1391104 - 23104000x_1 - 3258121461278087217393416979000000000000x_1^{25} + 349\\
&36653394018915017860319587726 40x_1^{19} + 652867238680135974158431761702720000000x_1^{23}\\
&- 19173223296x_1^2 - 14238685474284608x_1^5 + 169378340955991608x_1^6 - 52507859521071\\
&8729558176886016720x_1^{18} - 1901929311166539383292x_1^8 + 18808220394605923034922x_1^9\\
&+ 1951901333819610473221164x_1^{10} - 859885172496895877189725 68x_1^{11} + 969959260249\\
&293849600647844x_1^{12} + 28711240128464090390303244750x_1^{13} - 1212927424542577673412\\
&703176456x_1^{14} + 189916458460297701327904642236 00x_1^{15} - 1396883480747208689358908\\
&29032660x_1^{16} + 158990516835104x_1^4 + 1471039408752212735453693409497136 00x_1^{20} +\\
&372359132677171993047639226052760x_1^{17} + 33438688925144477112x_1^7 - 1103962286732\\
&41692628959355979500800 00x_1^{22} + 54764751031608728775885129000000000000000x_1^{26} +\\
&642275140992x_1^3
\end{aligned}
$$

$\tilde{L}_1$ has three true regular singularities $\{0, \frac{1}{24}, -\frac{1}{24}\}$, which will help us to find the hypergeometric solution of L in the later chapter. The length of $\tilde{L}_1$ to be 2200.

Next, we will use the improved algorithm to find $\tilde{L}$.

Step 1: The remainder of $\sigma(G)G$ right divided by L is a constant b, we compute this $b = \frac{1}{4}$.

Step 2: Take $\lambda = \frac{1}{\sqrt{b}} = 2$ (-2 is another option).

Step 3: Compute $\mathrm{LCLM}(1 + \lambda G, L)$ to obtain $\tilde{L}(1 + \lambda G)$. Right divide by $1 + \lambda G$ to find $\tilde{L}$, which is also sitting in $\mathbb{Q}(x_1)[\partial_{x_1}]$, we still use the variable x_1 to represent x^2 here:

$$
\begin{aligned}
\tilde{L}_2 :=&(185247046575000000 0x_1^{15} - 3133868014170000000x_1^{14} + 1609763031521606250x_1^{13} -\\
&425874593515515750x_1^{12} + 132740261271836595x_1^{11} - 41369947864562889x_1^{10} +\\
&6579339419282373x_1^9 - 444742282853595x_1^8 + 3133639552077x_1^7 + 1113115312041x_1^6\\
&- 46785562799x_1^5 - 140959783x_1^4 + 42396400x_1^3 - 520232x_1^2 - 10896x_1 + 208)x_1\partial^2 +
\end{aligned}
$$

$$2(6483646630125000000x_1^{15} - 8260896456961875000x_1^{14} + 3126521472447037500x_1^{13}$$

$$- 810828950512484250x_1^{12} + 402040898799896475x_1^{11} - 123637939965146322x_1^{10} +$$

$$15248640997597149x_1^{9} - 621601157188350x_1^{8} - 17573671307349x_1^{7} + 1858874046582$$

$$x_1^{6} - 9641820645x_1^{5} - 2263988288x_1^{4} + 53034286x_1^{3} + 423420x_1^{2} - 25416x_1 + 208)\partial +$$

$$2(8104558287656250000x_1^{14} - 6941568580410937500x_1^{13} + 1776968746215750000x_1^{12}$$

$$- 819387345274638750x_1^{11} + 556421023172069325x_1^{10} - 127645594966265238x_1^{9} +$$

$$9723104314831908x_1^{8} - 57017629317087x_1^{7} - 20820322303191x_1^{6} + 443420532966x_1^{5} +$$

$$17438903612x_1^{4} - 60688347x_1^{3} - 47088630x_1^{2} + 1654156x_1 - 17224)$$

$\tilde{L}_2$ has three true regular singularities $\{0, \frac{1}{24}, -\frac{1}{24}\}$ as well. Actually, $\tilde{L}_1$ is gauge equivalent to $\tilde{L}_2$. However, getting $\tilde{L}_2$ by the improved algorithm takes less CPU time than getting $\tilde{L}_1$ by using the original algorithm, and also $\tilde{L}_2$ has shorter length than $\tilde{L}_1$ (1315 VS. 2200).

In general, the improved algorithm is faster than the previous algorithm, especially for differential operators of large size. Moreover, the improved algorithm tends to produce a simpler descent. The algorithm from Chapter 3 had to choose an element from a 2-dimensional vector space. The algorithm in this Chapter does not make any choice other than $\pm\sqrt{\frac{1}{b}}$ because it computes an $\tilde{L}$ with an additive property, namely $V(\tilde{L}) \subseteq V(L_4)$.

4.3 Application for Higher Order Differential Operators

2-descent is not limited to second order linear differential equations. It can also be applied to higher order linear differential equations. For higher order equations, one can still define the type of a singularity, but it will involve more than just one exponent-difference, so the algorithm for finding the Möbius transformations needs to be adjusted. Once we have the subfield the differential operator is supposed to descend to, the algorithms for finding the 2-descent still work. However, the *equiv* program will be replaced by *Homomorphisms* in maple because *equiv* program is designed only for second order differential operators. In this section, two examples are presented. One of them is a fourth order differential operator, the 2-descent of which can be found by both the old algorithm and the improved algorithm; Another example is a third order differential operator, the 2-descent of which can be found by the improved algorithm, but not by the old algorithm in limited CPU time.

The following fourth order linear differential equation example comes from [2].

$$\begin{aligned}L :=&\partial^4 + \frac{(7x^4 - 68x^3 - 114x^2 + 52x - 5)}{(x+1)(x^2 - 10x + 1)(x-1)x}\partial^3 + \\ &\frac{2(5x^5 - 55x^4 - 169x^3 + 149x^2 - 28x + 2)}{(x+1)x^2(x^2 - 10x + 1)(x-1)^2}\partial^2 + \\ &\frac{2(x^4 - 13x^3 - 129x^2 + 49x - 4)}{(x+1)x^2(x^2 - 10x + 1)(x-1)^2}\partial - \\ &\frac{3(x+1)^2}{(x-1)^2x^3(x^2 - 10x + 1)}\end{aligned}$$

L has 4 regular true singularities:

$$p = 0, \infty, 1, -1$$

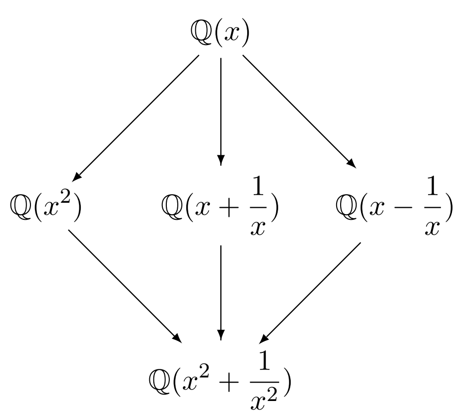
Among these 4 singularities, $0, \infty$ have the same type (at both points, the formal solutions involve the cube of a logarithm). At the singularities $1, -1$, the solutions also have a logarithm (but not a square or a cube of a logarithm). Hence $\sigma(\{0, \infty\})$ must be $\{0, \infty\}$ and $\sigma(\{-1, 1\})$ must be $\{-1, 1\}$. Then we find the set of Möbius transformations with order 2 as follows:

$$M_C^{\text{type}} = \{-x, \tfrac{1}{x}, \tfrac{-1}{x}\}$$

Here, $C = \mathbb{Q}$. For these 3 Möbius transformations, we find 3 subfields $\mathbb{Q}(x^2)$, $\mathbb{Q}(x + \frac{1}{x})$ and $\mathbb{Q}(x - \frac{1}{x})$ of index 2 respectively.

The possible 2-descent reductions for L:

Diagram 6



Next, take $\sigma = -x$ for example, we will show how to find $\tilde{L}$ defined over $\mathbb{Q}(x^2)$.

We compute the gauge transformation between L and $\sigma(L)$:

$$\begin{aligned} G := & \frac{x^3(x-1)^2(x^4+24x^3-18x^2+24x+1)}{(x+1)^4(x^2-10x+1)}\partial^3 + \\ & \frac{3x^2(x-1)(x^5+39x^4-26x^3+58x^2-7x-1)}{(x+1)^4(x^2-10x+1)}\partial^2 + \\ & \frac{x^6+88x^5-65x^4+240x^3-65x^2-8x+1)x}{(x^4-8x^3-18x^2-8x+1)(x+1)^2}\partial + \\ & \frac{x^3+9x^2-9x-1}{2(x^3-9x^2-9x+1)} \end{aligned}$$

Since $L \sim_g \sigma(L)$ we can compute the 2-descent $\tilde{L}$ by both the algorithm "Case A" and the improved algorithm "Improved Case A". We will show both of the computation in the following:

Case A

We follow the steps of the algorithm in Section 4.1.

Step 1, set $A := (a_{30} + a_{31}x)\partial^3 + (a_{20} + a_{21}x)\partial^2 + (a_{10} + a_{11}x)\partial + a_{00} + a_{01}x$.

Step 2, compute $A - \sigma(A)\lambda G$ right divided by L, set the remainder to be 0, we get 8 equations in 8 unknowns a_{ij}. Let M be the corresponding 8×8 matrix.

Step 3, compute the determinant of M, we find an equation of λ: $(\lambda-2)^4(\lambda+2)^4R(x^2)$, here $R(x^2) \in \mathbb{Q}(x^2)$. We solve for λ and find $\lambda = \pm 2$. We choose $\lambda = 2$ and find

$$\begin{aligned} A := & \frac{(3+3x^8-12x^6+18x^4-12x^2)}{6(5x^4+10x^2+1)(x^2+3)}\partial^3 + (1 + \frac{1+3x^8-42x^6-52x^4-38x^2}{2x(5x^4+10x^2+1)(x^2+3)})\partial^2 + \\ & (\frac{3x^{10}-135+414x^6-273x^2+90x^4-99x^8}{6(x^4+2x^2-3)(5x^6+5x^4-9x^2-1)} - \frac{-27x^8+132x^6+6x^4-108x^2-3}{6x(5x^8-14x^4+8x^2+1)})\partial \end{aligned}$$

Note 3: The kernel of $M - \lambda$ is a 4-dimensional $\mathbb{Q}(x)$-vector space, and any nonzero element in it provides an equally valid A. This corresponds to the non uniqueness of $\tilde{L}$.

Finally, we found 2-descent $\tilde{L}$ of L in $\mathbb{Q}(x^2)[\partial]$, which is written by new variable x_1 with $x_1 = x^2$:

$$
\begin{aligned}
\tilde{L}_{x_1} :=& 16x_1^4(x_1+3)(5x_1^2+10x_1+1)(9x_1^8+1008x_1^7-31820x_1^6+264480x_1^5 \\
&-14194x_1^4+162992x_1^3-8156x_1^2+18368x_1+529)(x_1-1)^4\partial^4 \\
&+32x_1^3(-7935-358000x_1-3502550x_1^2-24264785x_1^4-1520720x_1^3 \\
&-12737440x_1^5-13562976x_1^7-20800372x_1^6-905046x_1^{10}+20706063x_1^8 \\
&+28080x_1^{11}+6593808x_1^9+225x_1^{12})(x_1-1)^3\partial^3 \\
&+8x_1^2(2250x_1^{13}+312135x_1^{12}-12439492x_1^{11}+134614866x_1^{10} \\
&-42449802x_1^9-470021643x_1^8+267358792x_1^7-102361428x_1^6+163767350x_1^5 \\
&+221768417x_1^4-11134724x_1^3+48114210x_1^2+3717898x_1+77763)(x_1-1)^2\partial^2 \\
&+8x_1(x_1-1)(1350x_1^{14}+230355x_1^{13}-10741153x_1^{12}+169118578x_1^{11} \\
&-503407892x_1^{10}+340703465x_1^9+768939585x_1^8-411403540x_1^7 \\
&+839007558x_1^6-333028107x_1^5-52500447x_1^4+44391810x_1^3-43359960x_1^2 \\
&-2602385x_1-42849)\partial \\
&+720x_1^{15}+210495x_1^{14}-9498286x_1^{13}+240224513x_1^{12}-1412138412x_1^{11} \\
&+4365382207x_1^{10}-7520009378x_1^9-2959167271x_1^8-2667880856x_1^7 \\
&-5367819659x_1^6-136668050x_1^5-365681445x_1^4-305688780x_1^3+30068365x_1^2 \\
&+2524194x_1+14283
\end{aligned}
$$

By intersecting the set of singularities of $\tilde{L}_{x_1}$ and of $\mathrm{LCLM}(\tilde{L}_{x_1}, \partial_{x_1})$, we see that the set of true singularities of $\tilde{L}_{x_1}$ is $\{0, 1, \infty\}$. By observing the exponents at these 3 points, we can guess that $\tilde{L}_{x_1}$ has ${}_4F_3$ type solutions. We check this guess with DEtools[Homomorphisms] and also get the ${}_4F_3$ type solution of L in this way, see [13] for details.

Improved Case A

In this part, we follow the algorithm in previous section.

Step 1, Compute $\sigma(G)$, the remainder of $\sigma(G)G$ right divided by L is a constant b, we compute this constant $b = \frac{1}{4}$.

Step 2, Take $\lambda = \frac{1}{\sqrt{b}} = 2$ (-2 is another option).

Step 3, Compute LCLM$(1+\lambda G, L)$ to obtain $\tilde{L}(1+\lambda G)$. Right divide by $1+\lambda G$ to find $\tilde{L}$, which is also sitting in $\mathbb{C}(x^2)[\partial]$, we still use the normal variable x_1 to represent x^2 here:

$$\begin{aligned}\tilde{L_{1x_1}} :=& \partial^4 + \frac{77x_1^6 - 1709x_1^5 - 11250x_1^4 - 11530x_1^3 + 10377x_1^2 - 2457x_1 + 108}{(x_1-1)x_1(11x_1^5 - 215x_1^4 - 1250x_1^3 - 1278x_1^2 + 711x_1 - 27)}\partial^3 + \\ & \frac{220x_1^7 - 6063x_1^6 - 46066x_1^5 - 40985x_1^4 + 71024x_1^3 - 30225x_1^2 + 3078x_1 - 135}{2(x_1^2 - 2x_1 + 1)x_1^2(11x_1^5 - 215x_1^4 - 1250x_1^3 - 1278x_1^2 + 711x_1 - 27)}\partial^2 + \\ & \frac{22x_1^6 - 931x_1^5 - 10011x_1^4 - 12590x_1^3 + 15680x_1^2 - 3039x_1 + 117}{(x_1^2 - 2x_1 + 1)x_1^2(11x_1^5 - 215x_1^4 - 1250x_1^3 - 1278x_1^2 + 711x_1 - 27)}\partial - \\ & \frac{3(121x_1^5 + 175x_1^4 - 166x_1^3 + 1118x_1^2 - 227x_1 + 3)}{16(x_1^2 - 2x_1 + 1)x_1^4(11x_1^4 - 248x_1^3 - 506x_1^2 + 240x_1 - 9)}\end{aligned}$$

$\tilde{L_{1x_1}}$ also has three true regular singularities $\{0, 1, \infty\}$, Same as $\tilde{L}_{x_1}$, we can also find the ${}_4F_3$ type solution of L in terms of the solution of $\tilde{L_{1x_1}}$.

We compare the computation of these two algorithms and find that both algorithms give us nice results. However, the CPU time of the improved algorithm is less than the original algorithm. Also, the length of $\tilde{L_{1x_1}}$ is 635 which is shorter than the length of $\tilde{L}_{x_1}$ with 1080, which speeds up the computation of their hypergeometric solutions (the topic of Chapter 5).

The 2-descent algorithm can also be applied to third order differential operators:

$$\begin{aligned}L :=& 3x^2(x-1)(5x-1)(4x+1)(3x+1)(4x-1)(16x^2+1)(2x-1)^3(214412820480x^{16} + \\ & 88785027072x^{15} + 285633675264x^{14} + 284788850688x^{13} + 147070844928x^{12} + \\ & 114581954560x^{11} + 51090012160x^{10} + 17924375232x^9 + 3547140288x^8 + 648475792x^7 \\ & + 87537568x^6 + 13777552x^5 + 1569564x^4 + 94964x^3 + 6785x^2 + 7x - 4)\partial^3 +\end{aligned}$$

$$x(2x-1)^2(-76-47131072304x^8+1448683614064640x^{15}-613999773351936x^{16}+ 558927118911488x^{14}+18917847105536x^{12}-57777188555980800x^{23}+ 92859308588672x^{13}-261437033408x^9-6861938291671040x^{17}+82334523064320000 x^{24}+636x+289279x^4-1116375620x^6-309547373472x^{10}+96872577191903232x^{22}- 6931213924x^7+251437x^3-2922154920247296x^{21}-75903825x^5-56076956998828032 x^{20}-382098270624x^{11}+7266970890141696x^{19}-19960928256655360x^{18}+89101x^2)\partial^2+$$

$$2(2x-1)(42-164733425514112x^{14}+542953828x^7+4958782856183808x^{17}+51655341 2x^6+13654978x^4+88274057x^5+468530151700480x^{15}+13574817550831616x^{16}+ 7938858033610752x^{18}+378738806095872000x^{25}-370975395010314240x^{24}-532x+ 614766332960833536x^{23}+779481x^3-28835162990304x^{12}+161978322335563776x^{20}- 5421251819504x^{11}-265449451743608832x^{21}-2102081792240x^{10}- 132961869294993408x^{22}-127565903402176x^{13}-219975800720x^9-7594x^2-268267913 68x^8-128756752067854336x^{19})\partial+$$

$$2(-42+822089881567100928x^{20}-108284x^2+86052887x^4+50815388553347072x^{16}- 746638764675170304x^{22}+40457986512x^7-8698569821650944x^{17}+7663479184x^6-73 30x+589950026x^5+209100374400x^8+922146658320384000x^{25}-741370413975601152 x^{21}-584062130062360576x^{19}+159863982109491200x^{18}-1193304779988664320x^{24}- 18516469065879552x^{15}+1998109551643066368x^{23}+3545939x^3-1289881326127104x^{14}+ 48693630623040x^{12}-1154208533965952x^{13}-21793466677984x^{11}+279526241680x^{10}+ 283473002608x^9)$$

This is a huge linear operator with length 2686. Due to the limitation of the page space, the computation of the 2-descent is displayed in [13]. In this example both the 2-descent algorithms work. After analyzing the singularity structure, we found that his example involves 2 rounds of 2-descent, both of these two rounds belong to Case A. Our improved algorithm gave us a less complex first round 2-descent $\tilde{L}$ with a length of 3115. On the other hand, the old algorithm produced a $\tilde{L}$ with length 11857. More interesting, the improved algorithm found a final 2-descent with 3 true regular singularities $\{0, \infty, \frac{1}{256}\}$ which will help us find the hypergeometric solution of L. However, the old algorithm did not produce any result because the $\tilde{L}$ of length 11857 was too large to handle. The following will show the result gotten from the new algorithm.

We compute the singularities: $\{0, \infty, \frac{1}{4}, -\frac{1}{4}, RootOf(x^2 + \frac{1}{16})\}$. After comparing their singularities structure, we found the Möbius transformations set as $\{-x\}$. Apply the improved algorithm to L, we computed the constant $c = \frac{1}{9}$ and then took $\lambda = 3$, we finally found the first round of 2-descent as follows (Here $x_1 = x^2$):

$$
\begin{aligned}
\tilde{L}_{x_1} :=\,& 24(9x_1-1)(4x_1-1)x_1^3(x_1-1)(16x_1-1)(25x_1-1)(16x_1+1)(1793782427575910\\
& 400x_1^{14} - 1643131946359848960x_1^{13} + 912748687492055040x_1^{12} - 405864704917757\\
& 952x_1^{11} + 81349312593244416x_1^{10} + 6283677216516864x_1^9 - 1373341443512208x_1^8 -\\
& 195499004118816x_1^7 + 40385401814084x_1^6 - 2377302231583x_1^5 + 32081727473x_1^4 - 4\\
& 31018582x_1^3 + 80624374x_1^2 + 48229x_1 + 21)\partial^3 +\\
& 4x_1^2(268636856353768341504000000x_1^{20} - 524245378373704847720448000x_1^{19} + 460489\\
& 038133506052246732800x_1^{18} - 281352520760181933264076800x_1^{17} + 1251967251316321\\
& 2375982080x_1^{16} - 31455344488197306388316160x_1^{15} + 19112908553466265314816x_1^{14}\\
& + 5198953807655385678720x_1^{13} - 2123835153209662680768x_1^{12} - 2257867229628\\
& 918509488x_1^{11} + 410731343746448763296x_1^{10} - 29720555165638734704x_1^9 + 7895751\\
& 19423787437x_1^8 + 7217554368774676x_1^7 + 174819060613672x_1^6 - 41762810727548x_1^5 -\\
& 1253808426758x_1^4 + 94361582476x_1^3 - 496804000x_1^2 - 592516x_1 - 399)\partial^2 +\\
& \ldots
\end{aligned}
$$

we computed the singularity structure of $\tilde{L}_{x_1}$ and found the singularities set as $\{0, \infty, \frac{1}{16}, \frac{-1}{16}\}$ and the Möbius transformations set $\{-x\}$. The gauge transformation between $\tilde{L}_{x_1}$ and $\sigma(\tilde{L}_{x_1})$ also exists, that means there may be another round of 2-descent. We got the constant $c = \frac{8696601}{766846864}$ and took $\lambda = \frac{27692}{2949}$. It finally produced the 2-descent with three true singularities in the subfield $\mathbb{C}(x^4)[\partial_{x^4}]$ as follows (Here $x_2 = x^4$):

$$
\begin{aligned}
\tilde{L}_{x_2} :=&192(81x_2-1)(16x_2-1)x_2^3(x_2-1)(256x_2-1)(625x_2-1)(4457363797211875293265\\
&920000000x_2^{14}+4970260598069982994272092160000x_2^{13}+395739456445423394036 2\\
&1084037120x_2^{12}-1485942319243823025439126027468 8x_2^{11}+254305591369398967 36\\
&02589291776x_2^{10}-2033760933704239915021959124 80x_2^{9}+849102321456227816828 7\\
&435425x_2^{8}-2043399455872367045398707 00x_2^{7}+27407323077836966835339 56x_2^{6}-1\\
&8172014331390231832342x_2^{5}+5097545139036084690 5x_2^{4}+76207691332194700x_2^{3}-\\
&431118318161370x_2^{2}-48918288818x_2-5159484)\partial^3+\\
&16x_2^2(7671515343015792024736455720960000000000 0x_2^{19}+525379228275058725709\\
&33057431797760000000x_2^{18}+813326022483654690136213457078279208960000x_2^{17}-\\
&98205105359401237643781776521805511524352 0x_2^{16}+349478874075171765338071 8\\
&93415409506320384x_2^{15}-65411798958513496924199600103814900740096x_2^{14}+6617\\
&054262324834793393646871926881238784x_2^{13}-3941828107061412920169761225329\\
&00245616x_2^{12}-1483894868352250641103342801370632604 9x_2^{11}-3621830288256391\\
&842971045467846585 03x_2^{10}+561946981534160685902774180001690 7x_2^{9}-529910956\\
&2048068732893768154690 3x_2^{8}+283549451334446370876297379147x_2^{7}-56134735439\\
&0083375782779573x_2^{6}-2148290744200387280792981x_2^{5}+1234585830135207499843 9\\
&x_2^{4}-152987325725273792 56x_2^{3}-1967162004830656x_2^{2}+307613990294x_2+1289871\\
&00)\partial^2+\cdots
\end{aligned}
$$

CHAPTER 5

SOLVING SECOND ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS IN TERMS OF HYPERGEOMETRIC FUNCTIONS

For a given linear differential operator L over K, after 2-descent if it exists, we end up with another differential operator $\tilde{L}$. This time $\tilde{L}$ is defined over a subfield k and has the same order as L. Under 2-descent, $\tilde{L}$ is easier to solve because $\tilde{L}$ has fewer true singularities. The number of true singularities of $\tilde{L}$ is at most $n/2+2$ (n is the number of true singularities of L). When L has order 2 and the number of true singularities of $\tilde{L}$ drops to 3, we can find hypergeometric solutions of $\tilde{L}$, and use them to find the closed form solutions of L in terms of ${}_2F_1$ hypergeometic functions. Therefore in this chapter, we focus on solving second order linear homogeneous differential equations in terms of ${}_2F_1$ type solutions by 2-descent reduction.

5.1 The general hypergeometric Series

A series with the form $\sum_{i=0}^{\infty} x_i$ is called hypergeometric if x_{i+1}/x_i is a rational function of i.

Definition 25. *The Pochhammer symbol* $(a)_k$ *is defined as the factorial* $a(a+1)(a+2)\cdots(a+k-1)$.

With this definition, we have the formal definition of the general hypergeometric series:

Definition 26. *A generalized hypergeometric series is defined by*

$$ {}_pF_q\left(\begin{array}{c} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{array} \middle| \, x \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n n!} x^n \tag{5.1} $$

Note: To ensure the denominator on the right-hand side is not zero, we assume that none of the b_i is a non positive integer. For convenience, the left-hand side is also denoted as ${}_pF_q(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; x)$.

There are many fundamental functions can be written as the the generalized hypergeometric form.

Example 11.

1. $$ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = {}_0F_0(x), $$

2. $$ \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = {}_0F_1(-; \frac{1}{2}; -\frac{x^2}{4}), $$

3. $$ \frac{1}{1-x} = {}_1F_0(1; -; x), \qquad \text{with } |x| < 1. $$

4. $$ \sin^{-1} x = x {}_2F_1(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2), \qquad \text{with } |x| < 1. $$

For any ${}_pF_q$ hypergeometric series, we need consider the convergence properties.

Theorem 6. *The generalized hypergeometric series* ${}_pF_q\left(\begin{array}{c} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{array} \middle| \, x \right)$

1. *converges absolutely for all x if $p < q+1$,*
2. *diverges for all $x \neq 0$ if $p > q+1$,*
3. *converges absolutely for $|x| < 1$ if $p = q+1$,*
4. *diverges for $|x| > 1$ if $p = q+1$.*

Proof. Check by the ratio test from calculus. □

Remark 23. *From the above theorem we know, when $p \leq q+1$, the series converges for $|x| < 1$. The series defines a hypergeometric function when it converges.*

Theorem 7. *The generalized hypergeometric function ${}_pF_q$ satisfies the following differential equation:*

$$x\prod_{n=1}^{p}(x\frac{d}{dx}+a_n)y(x)=x\frac{d}{dx}\prod_{n=1}^{q}(x\frac{d}{dx}+b_n-1)y(x)$$

Proof. Plug the function $y(x) = {}_pF_q$ to both sides of the equation, the corresponding coefficients will become equal. □

5.2 Gauss hypergeometric equation

When $p = 2$ and $q = 1$ the generalized hypergeometric function becomes the Gauss Hypergeometric function ${}_2F_1(x)$:

$$ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| \, x\right) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n \tag{5.2}$$

The Gauss hypergeometric function (5.2) satisfies a second order differential operator

$$L := x(x-1)\partial^2 + [(a+b+1)x-c]\partial + ab \tag{5.3}$$

It has 3 singularities 0, 1 and ∞, and all of them are regular singularities. Then we can consider the indicial equation in chapter 2 or use the Maple command *gen_exp*, we get the exponents at each of these 3 singularities points:

$$\begin{array}{lll} x=0 & e_1=0 & e_2=1-c \\ x=1 & e_1=0 & e_2=c-a-b \\ x=\infty & e_1=a & e_2=b \end{array}$$

As stated in Chapter 2, we can always find a fundamental system of solutions of L at the singularities points. Consider $e_2 - e_1$ at $x = 0, 1, \infty$, if none of $1-c, c-a-b, b-a$ is integer value, we can write the fundamental system of L in terms of ${}_2F_1(x)$ at each of these 3 points (See [4]).

At $x=0$, we have

$$
\begin{aligned}
y_1(x) =& {}_2F_1(a,b;c;x)\\
=&(1-x)^{c-a-b}{}_2F_1(c-a,c-b;c;x)\\
=&(1-x)^{-a}{}_2F_1(a,c-b;c;\frac{x}{x-1})\\
=&(1-x)^{-b}{}_2F_1(a-c,b;c;\frac{x}{x-1})\\
y_2(x) =&x^{1-c}{}_2F_1(a-c+1,b-c+1;2-c;x)\\
=&x^{1-c}(1-x)^{c-a-b}{}_2F_1(1-a,1-b;2-c;x)\\
=&x^{1-c}(1-x)^{c-a-1}{}_2F_1(a-c+1,1-b;2-c;\frac{x}{x-1})\\
=&x^{1-c}(1-x)^{c-b-1}{}_2F_1(1-a,b-c+1;2-c;\frac{x}{x-1})
\end{aligned}
$$

At $x=1$, we have

$$
\begin{aligned}
y_1(x) =& {}_2F_1(a,b;a+b+1-c;1-x)\\
=&x^{1-c}{}_2F_1(a-c+1,b-c+1;a+b+1-c;1-x)\\
=&x^{-a}{}_2F_1(a,a-c+1;a+b+1-c;1-\frac{1}{x})\\
=&x^{-b}{}_2F_1(b-c+1,b;a+b+1-c;1-\frac{1}{x})\\
y_2(x) =&(1-x)^{c-a-b}{}_2F_1(c-b,c-a;c-a-b+1;1-x)\\
=&(1-x)^{c-a-b}x^{1-c}{}_2F_1(1-a,1-b;c-a-b+1;1-x)\\
=&(1-x)^{c-a-b}x^{a-c}{}_2F_1(1-a,c-a;c-a-b+1;1-\frac{1}{x})\\
=&(1-x)^{c-a-b}x^{b-c}{}_2F_1(c-b,1-b;c-a-b+1;1-\frac{1}{x})
\end{aligned}
$$

At $x=\infty$, we have

$$
\begin{aligned}
y_1(x) =&x^{-a}{}_2F_1(a,a-c+1;a-b+1;x^{-1})\\
=&x^{-a}(1-\frac{1}{x})^{c-a-b}{}_2F_1(1-b,c-b;a-b+1;x^{-1})\\
=&x^{-a}(1-\frac{1}{x})^{c-a-1}{}_2F_1(a-c+1,1-b;2-c;\frac{1}{1-x})\\
=&x^{-a}(1-\frac{1}{x})^{-a}{}_2F_1(a,c-b;a-b+1;\frac{1}{1-x})
\end{aligned}
$$

$$
\begin{aligned}
y_2(x) =& x^{-b}{}_2F_1(b, b-c+1; b-a+1; x^{-1}) \\
=& x^{-b}(1-\frac{1}{x})^{c-a-b}{}_2F_1(1-a, c-a; b-a+1; x^{-1}) \\
=& x^{-b}(1-\frac{1}{x})^{c-b-1}{}_2F_1(b-c+1, 1-a; 2-c; \frac{1}{1-x}) \\
=& x^{-b}(1-\frac{1}{x})^{-b}{}_2F_1(b, c-a; b-a+1; \frac{1}{1-x})
\end{aligned}
$$

According to [4], the above 24 solutions are essentially the same except when $c, c-a-b, a-b \notin \mathbb{Z}$. If one of exponent difference $1-c, c-a-b, b-a$ is integer value, there would be a logarithmic solution at that point. The corresponding logarithmic solutions are given in [32, 4].

5.3 Solutions of second order differential operator with 3 true singularities

Consider the following example:

Example 12.

$$L := x(x-1)(x+1)\partial^2 + (1-x)\partial + \frac{6}{25}x - \frac{1}{5}$$

L has 3 true singularities at $1, -1, \infty$, so L is among our target differential operators that may have ${}_2F_1$ type solutions. Then how to find those solutions? When we study L closer, we find out L actually has 4 singularities at $0, 1, -1, \infty$, the generalized exponents at these points are as follows:

```
>gen_exp(L,T, x=0);
```

$$[[0, 2, T = x]]$$

```
>gen_exp(L,T, x=1);
```

$$[[0, 1, T = x-1]]$$

```
>gen_exp(L,T, x=-1);
```

$$[[0, 0, T = x+1]]$$

```
>gen_exp(L,T, x=0);
```

$$[[-2/5, T = 1/x], [-3/5, T = 1/x]]$$

We can see 0 *is an apparent singularity. The number of counted singularities is 3. However, it is still necessary to take this apparent singularity account when we are trying to find the* ${}_2F_1$ *type solutions.*

L as in this example is a typical differential operator that we want to find its ${}_2F_1$ type solutions. Next, we will make it clear about the information we have regarding to the hypergeometric equations and our input operator L.

From section 5.2, we know that a hypergeometric function is a solution of a Gauss hypergeometric equation (5.2). This hypergeometic equation has:

(a) Three true regular singularities, located at $0, 1, \infty$.

(b) No apparent singularities.

In this chapter, our input differential operator L has

(a) Three true regular singularities, located say at $p_1, p_2, p_3 \in \mathbb{P}^1$.

(b) Any number of apparent singularities.

Our goal in this chapter is to solve L in terms of hypergeometric functions. Thus, to solve L in terms of hypergometric functions, we need to apply two types of transformations:

(a) A Möbius transformation (a change of variables) to move $p1, p2, p3$ to $0, 1, \infty$.

(b) A projective equivalence $\sim_p$ to eliminate all apparent singularities.

For the rest of this section, we will focus on handling these problems.

The general question in this section is how can we find the ${}_2F_1$ type solution for L. i.e How can we solve L in terms of the solution of a hypergeometric equation? If we find the ${}_2F_1$ solution of a hypergeometric differential operator L_1 which is projectively equivalent to L, then we also find the ${}_2F_1$ type solution of L. This suggests the goal of this chapter:

Goal: Solve $L \in \mathbb{C}(x)[\partial]$ in terms of ${}_2F_1$-function. But if L is solvable in terms of solution of L_1 which is projectively equivalent to L, then we also achieve our goal.

Example 13. *Consider the differential operator*

$$L_1 := 64(x-1)x\partial^2 + 16(3x-1)\partial + 1$$

which is a Gauss hypergeometric equation. We find its ${}_2F_1$ *solution as*

$$y_1 = {}_2F_1(-\frac{1}{8}, -\frac{1}{8}; \frac{1}{4}; x).$$

Consider another differential operator

$$L_2 := 64(x-1)x\partial^2 + 16(3x-1)\partial - 15$$

which is also a hypergeometric equation and has the solution as

$$y_2 = {}_2F_1(-\frac{5}{8}, \frac{3}{8}; \frac{1}{4}; x)$$

Now $L_1 \sim_p L_2$*, we spell out this projective equivalence with the following relations:*

$$y_2 = \sqrt{1-x}(y_1 - 8x\frac{d(y_1)}{dx})$$

$$y_1 = \sqrt{1-x}(y_2 - \frac{8}{5}x\frac{d(y_2)}{dx})$$

for all x *of absolute value less than 1.*

So if L can not be solved in terms of solutions of L_1, then there is no need to check if L can be solved in terms of any L_2 with $L_2 \sim_p L_1$.

This raises the question: How to classify Gauss hypergeometric equations up to projective equivalence?

Theorem 8. *Let* L_1, L_2 *be two Gauss hypergeometric differential operators. Assume the exponent difference set of* L_1 *at* $0, 1, \infty$ *is* $\{e_0, e_1, e_\infty\}$*, and the exponent difference set of* L_2 *at* $0, 1, \infty$ *is* $\{d_0, d_1, d_\infty\}$*. If*

1. $e_i - d_i \in \mathbb{Z}$ *for all* $i \in \{0, 1, \infty\}$
   and

2. $\sum_{i\in\{0,1,\infty\}}(e_i - d_i)$ *is an even integer,*

Then $L_1 \sim_p L_2$*.*

Proof. Assume $\{e_0-d_0, e_1-d_1, e_\infty-d_\infty\} \in \{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(0,1,1),(1,0,1)\}$. We treat the first case in the following computation (all other cases can be obtained from these by induction).

In the first case, when $(e_0-d_0, e_1-d_1, e_\infty-d_\infty) = (2,0,0)$. We start with L_1, with the exponent difference e_0, e_1, e_∞ at $0, 1, \infty$, reconstruct L_1 by e_0, e_1, e_∞, we have

$$L_1 := x(x-1)\partial^2 - (-2x + xe_0 + xe_1 + 1 - e_0)\partial + \frac{(e_0 - 1 + e_1 + e_\infty)(e_0 - 1 + e_1 - e_\infty)}{4}$$

Since $(e_0-d_0, e_1-d_1, e_\infty-d_\infty) = (2,0,0)$, then we have $d_0 = e_0 - 2$,$e_1 = d_1$ and $e_\infty = d_\infty$, After replacing d_0, d_1, d_∞ by the corresponding e_i value, we have:

$$\begin{aligned} L_2 :=& x(x-1)\partial^2 - (-2x + x(e_0-2) + xe_1 + 1 - (e_0-2))\partial + \\ &\frac{((e_0-2) - 1 + e_1 + e_\infty)((e_0-2) - 1 + e_1 - e_\infty)}{4} \end{aligned}$$

After applying the *equiv* to L_1 and L_2, we do get an equivalence operator between L_1 and L_2

$$\text{equiv}(L_1, L_2) = x(x-1)\partial - \frac{2xe_0 + e_\infty^2 x + 3xe_0^2 - x - e_1^2 x - 4e_0^2 + 2xe_1 + 2xe_2e_1}{4(1+e_0)}$$

We proves the conclusion. □

The theorem can also be proved by considering the so-called monodromy of L_1 and L_2, but to give this proof requires making introduction more background.

This theorem says that if we have two Gauss hypergeometric equations, we can verify if they are equivalent based on the difference of the exponent difference at the singularities $0, 1, \infty$.

Corollary 2. *Let L_1, L_2 be two Gauss hypergeometric differential operator. Assume the exponent difference set of L_1 at $0, 1, \infty$ is $\{e_0, e_1, e_\infty\}$, and the exponent difference set of L_2 at $0, 1, \infty$ is $\{d_0, d_1, d_\infty\}$. If $\frac{1}{2} + \mathbb{Z}$ appears in $\{e_0, e_1, e_\infty\}$ and $\{d_0, d_1, d_\infty\}$, then L_1 is projectively equivalent to L_2 if $e_i - d_i \in \mathbb{Z}$ for all $i \in \{0, 1, \infty\}$.*

Proof. Since $\frac{1}{2} + \mathbb{Z}$ appears in each ordered triple, so it must appear at the same place in each ordered triple, otherwise, it will violate the condition $e_i - d_i \in \mathbb{Z}$ for all $i \in \{0, 1, \infty\}$. We suppose it occurs at the first place, i.e., $e_0 = \frac{1}{2} + \mathbb{Z}$ and $d_0 = \frac{1}{2} + \mathbb{Z}$. If $\sum_{i \in \{0,1,\infty\}}(e_i - d_i)$

is an even integer, then by Theorem 8, we have $L_1 \sim_p L_2$.
Otherwise, if $\sum_{i\in\{0,1,\infty\}}(e_i - d_i)$ is an odd integer. then we can still use the similar method as in Theorem 8 to verify the conclusion, the only difference is the set for $(e_0 - d_0, e_1 - d_1, e_\infty - d_\infty)$ in Theorem 8 changes from $\{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(0,1,1),(1,0,1)\}$ to $\{(1,0,0),(0,0,1),(0,1,0)\}$ (all the other cases can be obtained by induction). Here, we present another way to prove this. Since exponent-difference is written up to $\pm$ sign. Therefore, if d_0, e_0 are both $\frac{1}{2} + \mathbb{Z}$ and $\sum_{i\in\{0,1,\infty\}}(e_i - d_i)$ is an odd integer, then we can rewrite d_0 or e_0 to $-\frac{1}{2} + \mathbb{Z}$, in this way, our difference becomes an even integer. By Theorem 8, $L_1 \sim_p L_2$. This proves this corollary. □

Next we would discuss how to solve a second order linear homogeneous differential operator with 3 true regular singularities in terms of Gauss hypergeometric functions.

5.3.1 Finding the equivalent Gauss hypergeometric equation

Given L, we can compute the exponent difference d_0, d_1, d_∞ at p_1, p_2, p_3, respectively. Having these information, we are able to decide that if there is one Gauss hypergeometric equation with the same exponent differences module $\mathbb{Z}$ as L at the three regular singularities $0, 1, \infty$. (This means the three exponent differences $d_0 + \mathbb{Z}, d_1 + \mathbb{Z}, d_\infty + \mathbb{Z}$ must be the exponent difference at $0, 1, \infty$, the order of these three may be switched.). The following lemma ensures us the possible singularities structure of the possible $L_{_2F_1}$.

Lemma 14. *Suppose L is projectively equivalent to a hypergeometric equation. suppose that the exponent-differences of L at $0, 1, \infty$ are d_0, d_1, d_∞. Let L_1 be a hypergeometric equation with exponent-differences: d_0, d_1, d_∞ and L_2 be a hypergeometric equation with exponent-differences: $d_0 + 1, d_1, d_\infty$. Then $L \sim_p L_1$ or $L \sim_p L_2$ (both are true if $\{d_0, d_1, d_\infty\} \bigcap \{\frac{1}{2} + \mathbb{Z}\} \neq \emptyset$).*

Note: The easiest case is when L has no apparent singularities, then L can be reduced to L_1 using just an exp-product.

Proof. Assume $L \sim_p L_0$, where L_0 is a hypergeometric equation, then L_0 must have the same exponent-difference as L module $\mathbb{Z}$, because $\sim_p$ preserve the exponent-difference module $\mathbb{Z}$. Since L_1, L_2 both have same exponent-difference as L and L_0 module $\mathbb{Z}$. that means the first hypothesis in Theorem 8 is satisfied. If L_0 has the exponent-differences: d_0, d_1, d_∞, then the second hypothesis of Theorem 8 is satisfied, then $L_0 \sim_p L_1$, i.e $L \sim_p L_1$. Otherwise, if L_0 has the exponent-differences: d_0+1, d_1, d_∞, then $L_0 \sim_p L_2$, i.e $L \sim_p L_2$. Particularly, if $\{d_0, d_1, d_\infty\} \bigcap \{\frac{1}{2}+\mathbb{Z}\} \neq \emptyset$, then $L_0 \sim_p L_1 \sim_p L_2$ by Corollary 2, i.e $L \sim_p L_1 \sim_p L_2$. $\square$

Example 14. *Let*

$$L := 4x^2(16x-1)\partial^2 + 12x(16x-1)\partial + 64x - 3$$

we compute its singularities and found that it has three true regular singularities $0, \frac{1}{16}, \infty$ *, and also we have* $\Delta(L,0) = 0$*,* $\Delta(L, \frac{1}{16}) = 0$*,* $\Delta(L,\infty) = 0$*. Based on these three exponent-differences, we can construct the Gauss hypergeometric equations* L_1, L_2 *as follows:*
Case 1: Set $\Delta(L_1,0) = 0$*,* $\Delta(L_1,1) = 0$*, and* $\Delta(L_1,\infty) = 0$ *and plug them in the formula we have*

$$L_1 := x(x-1)\partial^2 - (-2x+1)\partial + \frac{1}{4},$$

which is a hypergeometric equation with singularities $0, 1, \infty$ *and exponent-differences* $0, 0, 0$*, respectively.*
Case 2: Set $\Delta(L_1,0) = 0$*,* $\Delta(L_1,1) = 0$*, and* $\Delta(L_1,\infty) = 1$ *and similarly we have*

$$L_2 := x(x-1)\partial^2 - (-2x+1)\partial,$$

which is a hypergeometric equation with singularities $0, 1, \infty$ *and exponent-differences* $0, 0, 1$*, respectively.*

From the above theorem, we know that L_1 is not projectively equivalent to L_2. Check by using the *equiv* program:

```
>equiv(L_1,L_2)
```

$$0$$

Note: We can construct more than two such Gauss hypergeometric equations with the same exponent-difference module $\mathbb{Z}$ at $0, 1, \infty$. But all of them are projectively equivalent to L_1 or L_2.

Remark 24. *For a given second order differential operator L with three true regular singularities p_1, p_2, p_3, by Lemma 14, we know there are two Gauss hypergeometric equations L_1, L_2 (probably one) which have the same exponent-difference (module $\mathbb{Z}$)at $0, 1, \infty$ as at p_1, p_2, p_3. However, L_1, L_2 are not necessarily projectively equivalent to L after certain transformation.*

To check if L_1, L_2 are projectively equivalent to L, we first need find the transformation which sends p_1, p_2, p_3 to $0, 1, \infty$. Möbius transformation $m(x) = \frac{ax+b}{cx+d}$ would be such a transformation. (We can switch the order of p_1, p_2, p_3, the procedure would be the same). After the change of variable by $m(x)$ for L_1, L_2, we get two new differential operator L_1', L_2', both of them have the true regular singularities p_1, p_2, p_3. We will explain this by continuing example 14:
we compute the Möbius transformation $m(x)$, which sends $0, \frac{1}{16}, \infty$ to $0, 1, \infty$, we get

$$m(x) = 16x,$$

then we conduct the change of variable for L_1 and L_2 by $x \mapsto 16x$, we have

$$L_1' = x(16x-1)\partial^2 + (32x-1)\partial + 4$$

and

$$L_2' = x(16x-1)\partial^2 + (32x-1)\partial$$

We check by the *equiv* again to see if these two operators are projectively equivalent to L:

```
>equiv(L,L'_1);
```

$$x^{\frac{3}{2}}\partial + \frac{3}{2}\sqrt{x}$$

```
>equiv(L,L'_2);
```

$$0$$

We can see that after change of variable L_1 is projectively equivalent to L, while L_2 is not. That means L_2 would be discarded.

5.3.2 Algorithm

Once we have the appropriate Gauss hypergeometric equation, we now can compute the ${}_2F_1$ solution of the second order differential operator L with three true regular singularities.

Given the Gauss hypergeometric equation L_1 as found previously, which is projectively equivalent to L under certain Möbius transformation $m(x)$. That means we can find the solution space $V(L)$ by finding the solution space $V(L_1)$ of L_1, furthermore the solution space $V(L_1')$ of L_1'.

Given the exponent-differences of L_1, to find the ${}_2F_1$ solution we need find the corresponding a, b, c as in equation (5.2). Assume the exponent-difference at $0, 1, \infty$ are e_0, e_1, e_∞, respectively, then we compute the a, b, c by the information given previously, we get:

$$a = \frac{1}{2} - \frac{1}{2}e_0 - \frac{1}{2}e_\infty - \frac{1}{2}e_1$$

$$b = \frac{1}{2} - \frac{1}{2}e_0 + \frac{1}{2}e_\infty - \frac{1}{2}e_1$$

$$c = 1 - e_0$$

Now, if $1 - c$ is not an integer , then we have the two independent solutions as:

$$y_1(x) = {}_2F_1(a, b; c; x)$$

$$y_2(x) = x^{1-c}{}_2F_1(a - c + 1, b - c + 1; 2 - c; x)$$

and then the general solution would be

$$C_1 y_1(x) + C_2 y_2(x),$$

where $C_1, C_2 \in K$.

If $1-c$ is integer, the first solution is still the same as $y_1(x)$ and we can select the second solution as

$$y_2(x) = {}_2F_1(a, b; a + b + 1 - c; 1 - x)$$

which is also independent with $y_1(x)$. Under the Möbius transformation $m(x)$, we can write out the basis of the solution space $V(L_1')$: if $1 - c$ is not an integer , then we have the two independent solutions as:

$$y_1'(x) = {}_2F_1(a, b; c; m(x))$$

$$y_2'(x) = m(x)^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; m(x))$$

If $1-c$ is integer, we have the same $y_1'(x)$, but

$$y_2'(x) = {}_2F_1(a, b; a+b+1-c; 1-m(x)).$$

Since we can find the equivalence G between $V(L_1')$ and $V(L)$ by *equiv*, that means for any general solution $C_1 y_1'(x) + C_2 y_2'(x)$ of L_1', $G(C_1 y_1(x) + C_2 y_2(x))$ would be a solution of L.

We will show this procedure by continuing example 14. We have

$$L_1 := x(x-1)\partial^2 - (-2x+1)\partial + \frac{1}{4},$$

and

$$L_1' = x(16x-1)\partial^2 + (32x-1)\partial + 4,$$

which is projectively equivalent to L. Since L_1 has the exponent-difference $0, 0, 0$ at $0, 1, \infty$, respectively. so we have $a = \frac{1}{2}$, $b = \frac{1}{2}$ and $c = 1$, that means the two independent solutions would be

$$y_1(x) = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$$

$$y_2(x) = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)$$

Furthermore, we have:

$$y_1'(x) = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 16x)$$

$$y_2'(x) = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-16x)$$

the general solution of L_1' would be $C_1 y_1'(x) + C_2 y_2'(x)$. Next, we find the solution of L. First we compute G, which would be

$$G := \frac{16x-1}{\sqrt{x}}\partial$$

Then, we can compute the solution of L by the following commands:

```
>eval(DEtools[diffop2de](G, y(x)), y(x) = C_1*y_1(x)+C_2*y(x));
```

$$C_1\frac{4(16x-1)}{\sqrt{x}}{}_2F_1(\frac{3}{2},\frac{3}{2};2;16x)-C_2\frac{4(16x-1)}{\sqrt{x}}{}_2F_1(\frac{3}{2},\frac{3}{2};2;1-16x)$$

Algorithm finding ${}_2F_1$-type solution with 3 singularities

Input: A second order irreducible differential operator L with 3 true regular singularities.

Output: ${}_2F_1$-type solution, if it exists..

Step 1: Compute the exponent-difference at the three singularities of L module $\mathbb{Z}$. Denote them as e_1, e_2, e_3.

Step 2: Find the two Gauss hypergeometric equations L_1, L_2 by the formula, whose exponent difference at $0, 1, \infty$ module $\mathbb{Z}$ is e_1, e_2, e_3 if $\frac{1}{2} \notin \{e_1, e_2, e_3\}$; Otherwise, compute just one such equation L_1.

Step 3: Find the Möbius transformation $m(x)$ between p_1, p_2, p_3 and $0, 1, \infty$.

Step 4: Call **equiv** to check if L_1 (after change of variable) is projectively equivalent to L, if so, go to next step. Otherwise, check L_2. Or stop if both of them are not projective equivalence to L. Denote the equivalence as G

Step 5: Find the Gauss hypergeometric solutions of $Sol := C_1y_1(m(x)) + C_2y_2(m(x))$ if $e_1 \neq 0$, otherwise, compute $Sol := C_1y_1(m(x)) + C_2y_2'(m(x))$.

Step 6: Compute the ${}_2F_1$-type solution of L by computing $G(Sol)$.

5.4 Final solving after 2-descent

In this section, we assume our differential operator are all defined over $K = \mathbb{C}(x)$. From Chapter 3, we descended a second order differential operator L defined over K to another differential operator $\tilde{L}$ which is defined over $\mathbb{C}(f)$, where $\deg(f) = 2$, if it exists. This time $\tilde{L}$ is easier to solve as it will have few true singularities. Sometime we can reduce $\tilde{L}$ further to another even easier operator. If the new differential operator $\tilde{L}$ has three true regular singularities, we can find its ${}_2F_1$ solutions from previous section. Moreover, we can find the ${}_2F_1$ solutions of the original operator L.

5.4.1 Algorithm of finding ${}_2F_1$-type solution by 2-descent

We assume L descends to $\tilde{L}$ which have three true regular singularities. Actually, $\tilde{L}$ is written in the small field $\mathbb{C}(f)[\partial]$. For example, if $L \in \mathbb{C}(x)[\partial]$ descends to $\tilde{L} \in \mathbb{C}(x^2)[\partial]$, then we usually write $\tilde{L}$ in the new variable x_1, which is actually x^2. Only in this way, $\tilde{L}$ have few singularities than L. All the new singularities are about x_1.

Now, we assume $\tilde{L}$ is gotten after one round of 2-descent (Similar procedure can also applied to several rounds of 2-descent).

Since $\tilde{L}$ is defined over the subfield $\mathbb{C}(f)$ of $\mathbb{C}(x)$, that means after the transformation $x \mapsto f$, we get a new differential operator $\tilde{L'}$ which is defined over $\mathbb{C}(x)$. From Chapter 3, we also know that $\tilde{L'}$ is projectively equivalent to L, that means we can find the solution space $V(L)$ by computing $V(\tilde{L'})$. The Algorithm of finding the final solutions of L is as follows:

Algorithm of finding ${}_2F_1$-type solution by 2-descent

Input: A second order irreducible differential operator $L \in C(x)[\partial]$ and the field C.

Output: ${}_2F_1$-type solution, if it exists..

Step 1: Call **Algorithm 2-descent** in Chapter 3 to Compute the 2-descent of L, $\tilde{L}$, if it exists.

Step 2: Compute the true singularities of $\tilde{L}$.

Step 3: If $\tilde{L}$ has 3 true regular singularities, then call **Algorithm finding ${}_2F_1$-type solution with 3 singularities** and find the solution *sol*; Otherwise, stop and return NULL.

Step 4: Apply the Change of variable $x \mapsto f$ to $\tilde{L}$, Sol, we get $\tilde{L'}$ and its ${}_2F_1$ solution Sol'.

Step 5: Call **equiv** to Compute the equivalence G between $\tilde{L'}$ and L.

Step 6: Compute the ${}_2F_1$-type solution of L by computing $G(Sol')$.

Remark 25. *We may have the descent $\tilde{L}$ with 3 true regular singularities by conducting more than one rounds of 2-descent. If so, to compute the ${}_2F_1$-type solution of L, we only need repeat Step 4 and Step 5 in this algorithm.*

5.4.2 Examples

In this section, we would compute the ${}_2F_1$-type solutions of the examples listed in Chapter 3.

Example 15. *Let*

$$L = \partial^2 + \frac{28x-5}{x(4x-1)}\partial + \frac{144x^2+20x-3}{x^2(4x-1)(4x+1)}$$

Step 1: Compute the 2-descent of L from Section 3.7, we have

$$\tilde{L} := (16x-1)x\partial^2 + (32x-2)\partial + 4$$

step 2: Compute the true singularities of $\tilde{L}$, we found it has 3 true regular singularities: $0, \frac{1}{16}, \infty$.

step 3: Call Algorithm finding ${}_2F_1$-type solution with 3 singularities, we found the ${}_2F_1$ solution of $\tilde{L}$ as

$$Sol := C_1(64x-4){}_2F_1(\frac{3}{2},\frac{3}{2};2;16x) - C_2(64x-4){}_2F_1(\frac{3}{2},\frac{3}{2};2;1-16x)$$

Step 4: From Section 3.7, we know that $f = x^2$, so the change of variable would be $x \mapsto x^2$. Apply transformation to $\tilde{L}$ and Sol, we have

$$\tilde{L}' := x(4x+1)(4x-1)\partial^2 + (12x-3)(4x+1)\partial + 16x$$

$$Sol' := C_1(64x^2-4){}_2F_1(\frac{3}{2},\frac{3}{2};2;16x^2) - C_2(64x^2-4){}_2F_1(\frac{3}{2},\frac{3}{2};2;1-16x^2)$$

Step 5: Compute the equivalence between $\tilde{L}'$ and L, we have

$$G := \frac{1}{x(4x-1)}$$

Step 6: Compute $G(Sol')$, we have the final solution as

$$C_1\frac{16x+1}{x}{}_2F_1(\frac{3}{2},\frac{3}{2};2;16x^2) - C_2\frac{16x+1}{x}{}_2F_1(\frac{3}{2},\frac{3}{2};2;1-16x^2)$$

Example 16. *Consider the example 9*

$$L := \partial^2 + \frac{4(1296x^5+576x^4-144x^3-72x^2+x+1)}{x(6x-1)(2x+1)(6x+1)(12x^2-1)}\partial + \frac{2(5184x^6-864x^5-1656x^4+48x^3+162x^2+6x-1)}{(-1+2x)x^2(6x-1)(2x+1)(6x+1)(12x^2-1)}$$

We apply the same procedures to get its ${}_2F_1$-type solution.

Step 1: After 2 rounds of 2-descent as in Section 3.7, we have

$$\begin{aligned}\tilde{L} :=& 4(36x+11)(18x-5)(6x+1)(6x-1)^2\partial^2 + \\ & 36(6x-1)(1296x^3+1620x^2+20x-9)\partial + \\ & 34992x^3 - 207036x^2 - 2331 + 3456x\end{aligned}$$

This time $\tilde{L}$ is defined over the subfield $\mathbb{C}(f)$, where $f := x^2 + \frac{1}{144x^2}$.

Step 2: Compute the singularities of $\tilde{L}$, we have 3 true regular singularities $\frac{5}{18}, -\frac{1}{6}, \infty$. For these 3 regular singularities, we have their exponent-difference module $\mathbb{Z}$ as $0, \frac{1}{2}, 0$, respectively.

Step 3: Call Algorithm finding ${}_2F_1$-type solution with 3 singularities. We can see $\frac{1}{2}$ is one of the exponent-difference, that means there is only one possible Gauss hypergeometric equation which is projectively equivalent to $\tilde{L}$. We found its

mbox${}_2F_1$ solution:

$$
\begin{aligned}
Sol := C_1(&\frac{-3\sqrt{6x-1}(90x-1)}{10(18x-5)}{}_2F_1(\frac{1}{4},\frac{1}{4};\frac{3}{2};\frac{9}{4}x+\frac{3}{8})- \\
&\frac{3\sqrt{6x-1}(6x+1)}{160}{}_2F_1(\frac{5}{4},\frac{5}{4};\frac{5}{2};\frac{9}{4}x+\frac{3}{8}))+ \\
C_2(&\frac{-54\sqrt{6x-1}(6x+1)}{5(18x-5)\sqrt{36x+6}}{}_2F_1(\frac{-1}{4},\frac{-1}{4};\frac{1}{2};\frac{9}{4}x+\frac{3}{8})- \\
&\frac{9\sqrt{6x-1}(6x+1)}{40\sqrt{36x+6}}{}_2F_1(\frac{3}{4},\frac{3}{4};\frac{3}{2};\frac{9}{4}x+\frac{3}{8}))
\end{aligned}
$$

Step 4: Apply 2 rounds of change of variable $x \mapsto x^2$ and $x \mapsto x + \frac{1}{144x}$, we finally have

$$
\begin{aligned}
\tilde{L'} :=& 4x^2(6x-1)(2x+1)(-1+2x)(6x+1)(44x^2+144x^4+1)(12x^2-1)^2\partial^2+ \\
& 8(152064x^8-1920x^6+1360x^4+96x^2-1)x(12x^2-1)\partial+ \\
& 2985984x^12-17667072x^10+357120x^8-444288x^6+2480x^4-852x^2+1
\end{aligned}
$$

We skip the solution of $\tilde{L'}$. To get its solution, we just replace x by $x^2 + \frac{1}{144x^2}$ in Sol.

Step 5: Compute equivalence between $\tilde{L'}$ and L, we have

$$
\begin{aligned}
G :=& \frac{(1-6x+8x^2-72x^3+144x^4)(24x^3+4x^2-6x-1)}{8(12x^2-1)^2(44x^2+144x^4+1)\sqrt{x}}\partial- \\
& \frac{(20736x^8+17280x^7-16128x^6-2208x^5+864x^4-184x^3-112x^2+10x+1)(-1+2x)}{16(44x^2+144x^4+1)x^{\frac{3}{2}}(12x^2-1)^3)}
\end{aligned}
$$

Step 6: Compute $G(Sol^{'})$, we finally solve L in terms of ${}_2F_1$ function as:

$$
\begin{aligned}
Sol := C_1(&\frac{-\sqrt{6}(12x^2-16x+1)}{160x^{\frac{5}{2}}(6x-1)}{}_2F_1(\frac{1}{4},\frac{1}{4};\frac{3}{2};\frac{1}{64}\frac{144x^4+1+24x^2}{x^2})-\\
&\frac{\sqrt{6}(2x-1)(12x^2+1)^2}{61440x^{\frac{9}{2}}}{}_2F_1(\frac{5}{4},\frac{5}{4};\frac{5}{2};\frac{1}{64}\frac{144x^4+1+24x^2}{x^2}))+\\
C_2(&\frac{\sqrt{6}}{20x^{\frac{3}{2}}(6x-1)}{}_2F_1(\frac{-1}{4},\frac{-1}{4};\frac{1}{2};\frac{1}{64}\frac{144x^4+1+24x^2}{x^2})-\\
&\frac{\sqrt{6}(2x-1)(12x^2+1)^2}{2560x^{\frac{7}{2}}}{}_2F_1(\frac{3}{4},\frac{3}{4};\frac{3}{2};\frac{1}{64}\frac{144x^4+1+24x^2}{x^2}))
\end{aligned}
$$

The following example from the combinatorics. Once we have the ${}_2F_1$ solution, we can find the generating function of that problem.

Example 17. *Let*

$$
\begin{aligned}
L :=& x^2(x-1)(3x-1)(5x-1)(15x^2-12x+2)^2(24x^2-1)(24x^2+1)(293760x^7-\\
&131976x^6+52704x^5-768x^4-5934x^3+1536x^2-141x+4)\partial^2+\\
&x(15x^2-12x+2)(456855552000x^{16}-1141140441600x^{15}+1205298040320x^{14}-\\
&739711751040x^{13}+278147198592x^{12}-42328568520x^{11}-16507230408x^{10}+\\
&12016561464x^9-3566060748x^8+612793944x^7-50926704x^6-4668138x^5+\\
&2379468x^4-404802x^3+38052x^2-1920x+40)\partial+\\
&17132083200000x^{18}-51619498560000x^{17}+69182207308800x^{16}-55259228764800x^{15}\\
&+28548905746176x^{14}-8578828641024x^{13}+203730145704x^{12}+1176651503856x^{11}-\\
&628573505184x^{10}+187319970144x^9-37321612578x^8+5110040592x^7-430267164x^6\\
&+5324838x^5+4336320x^4-663780x^3+50760x^2-2208x+48
\end{aligned}
$$

For this long equation, we compute its singularity structure as:

$$S_C^{type} := \{(x,0),(x^2-\frac{1}{24},0),(x^2+\frac{1}{24},0)\}$$

and then we compute its only Möbius transformation as $-x$ such that S_C^{type} is invariant under $-x$. We apply for the left part of the 2-descent program and get the descent operator

$\tilde{L}$ *as (we still use variable x instead of x_1 but be aware that x actually represent x^2):*

$\tilde{L}$:=$4x^2(24x-1)(24x+1)(146307465x^4 - 559059678x^5 - 9913872x^3 - 741616803x^6 -$
$77365538100x^7 - 4160x + 273551782500x^8 + 305032x^2 + 16(730950266336256000000$
$x^{14} + 283148327600225280000x^{13} + 30756872329502227200x^{12} + 1686893870166480576$
$x^{11} - 133823109463151832 * x^{10} - 28232747654035032x^9 + 1106153331674814x^8 +$
$245685032190468x^7 - 26914802257107x^6 + 1279546596900x^5 - 35033084322x^4 +$
$574811724x^3 - 5335541x^2 + 23464x - 32)(225x^2 - 84x + 4)^2\partial^2 +$
$4x(225x^2 - 84x + 4)(6144 + 184280985322333704840491757951 12x^{17} - 5757952x +$
$2252752704x^2 - 26684168933564011851610634474169600 0x^{22} +$
$6659516745167529844009013308454016 0x^{21} + 262089070044863840499620806 2x^{13} -$
$2122312046834711950349822 55x^{12} + 1037148412199553711954435 6x^{11} -$
$2902550301297311463420 21x^{10} + 1895854709589052854474x^9 + 21669638418794553$
$2469x^8 - 110574042141319574 76x^7 + 296895554744808636x^6 - 5202269056218912$
$x^5 + 103655504705828725934604288000000000 00x^{26} - 846468482231169224147376 69$
$5952x^{18} + 61390742993968x^4 - 475387061184x^3 - 152174484058985517174409864 39$
$50x^{16} + 9268877288296064419832514542 4x^{15} - 181279635624534245633944971 03$
$x^{14} + 585517501019365906538514682022400000x^{23} - 214338784196356831349508096$
$0000000000x^{25} - 11365747464304153256401621264040 64x^{19} - 2014113521485588153$
$31974581524640x^{20} - 13913081179850500256389680691200000 00x^{24})\partial +$
$244429964171626209036656217051236 4x^{17} - 14469120x + 24576 - 2765727782766074$
$7633448230634738863 36x^{21} + 94849160322224272541478887736x^{13} - 55482750391443$
$98932873564299x^{12} + 23108735006518240825363276 2x^{11} - 657278323540931448302 28$
$21x^{10} + 1116000717532917357060 00x^9 - 268607171394918291024x^8 - 4383900087965$
$5673664x^7 + \cdots$

Part of the coefficients for the ∂^0 term is skipped. We found the singularities structure of $\tilde{L}$ *as:*

$$S_C^{type} := \{(x,0),(x-\frac{1}{24},0),(x+\frac{1}{24},0)\}.$$

In the following, we compute the $_2F_1$ solution for $\tilde{L}$ and furthermore the final $_2F_1$ solution of L as follows:

$$C_1\Big(\frac{468x^4+162x^3-216x^2+43x-2}{\sqrt{24x^2+1}\,x(24x^2-1)(x-1)(3x-1)(5x-1)(15x^2-12x+2)}\,{}_2F_1(\frac{1}{2},\frac{1}{2};3;\frac{48x^2}{24x^2+1})+$$
$$\frac{2x(1926x^4-216x^3-216x^2+43x-2)}{(24x^2+1)^{3/2}(24x^2-1)(x-1)(3x-1)(5x-1)(15x^2-12x+2)}\,{}_2F_1(\frac{3}{2},\frac{3}{2};4;\frac{48x^2}{24x^2+1})\Big)$$
$$C_2\Big(\frac{9576x^4-216x^3-1485x^2+281x-16}{\sqrt{24x^2+1}\,x^3(x-1)(3x-1)(5x-1)(15x^2-12x+2)}\,{}_2F_1(\frac{1}{2},\frac{1}{2};3;-\frac{24x^2-1}{24x^2+1})-$$
$$\frac{(24x^2-1)(1926x^4-216x^3-216x^2+43x-2)}{3(24x^2+1)^{3/2}(x-1)(3x-1)(5x-1)(15x^2-12x+2)x^3}\,{}_2F_1(\frac{3}{2},\frac{3}{2};4;-\frac{24x^2-1}{24x^2+1})\Big)$$

CHAPTER 6

CONCLUSION

6.1 Contribution of this thesis

In this thesis, we developed 2-descent method for a linear differential operator L defined over $\mathbb{C}(x)$. We restrict our algorithm to second order differential operator. By using 2-descent, the number of true singularities of L will be reduced to no more than $n/2+2$ (n is the number of true singularities of L). To find the 2-descent of L, we have 2 steps to go:

1. Finding the subfield $\mathbb{C}(f)$ with $[\mathbb{C}(f):\mathbb{C}(x)]=2$, i.e. finding $f \in \mathbb{C}(x)$ of degree 2.

2. Finding the projectively equivalent differential operator $\tilde{L} \in \mathbb{C}(f)[\partial_f]$.

To realize the first task, we found the Möbius transformation σ, which satisfies the following conditions:

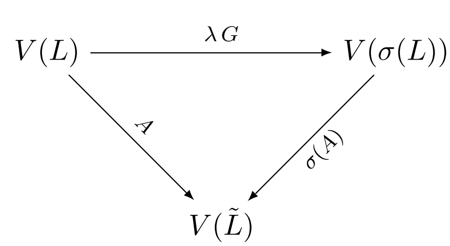
- $\sigma = \frac{ax+b}{cx+d}$ with $d=-a$;

- σ should preserve the set of true singularities of L and their exponent-difference mod $\mathbb{Z}$.

Once we have σ, we find the corresponding subfield $\mathbb{C}(f)$ by taking $f = x + \sigma(x)$ or $f = x \times \sigma(x)$.

To realize the second task, we construct a commutative diagram

Diagram 7



From this diagram, we see that $A - \sigma(A)\lambda G$ becomes a map from $V(L)$ to $V(\tilde{L})$, and has a nonzero kernel. This kernel corresponds to a right hand factor of L, since L is irreducible, the kernel is $V(L)$ itself. This fact enable us to construct equations about the unknowns in A, furthermore, we find the descent $\tilde{L}$.

In this thesis, we apply our 2-descent method to solve L in terms of Gauss hypergeometric solution if $\tilde{L}$ has 3 true regular singularities. To find the ${}_2F_1$ solution, we are finding the solutions with the form:

$$
\begin{aligned}
y_1 =& r_1 \cdot {}_2F_1\left(\begin{array}{c} a_1, a_2 \\ a_3 \end{array} \middle| \, f \right) + r_2 \cdot {}_2F_1\left(\begin{array}{c} b_1, b_2 \\ b_3 \end{array} \middle| \, f \right) \\
& \text{(with } r_1, r_2 \in \mathbb{C}(x)) \\
y_2 =& \cdots
\end{aligned}
$$

In the above formula, f is gotten from our 2-descent, it would be $x + \sigma(x)$ or $x \times \sigma(x)$ if there is only one round of 2-descent. Otherwise, it would be the combination of several f's. The parameters a_i, b_i for $i = 1, 2, 3$ are gotten from the connecting between the exponent-difference of the singularities of $\tilde{L}$ with the Gauss hypergeometric equation. r_1, r_2 are calculated from the projective equivalence, which have the form:

$$e^{\int r\,dx} \cdot (r_0 y + r^{'} y')$$

with $r, r_0, r^{'} \in \mathbb{C}(x)$. This is computed by the **equiv** program.

In this thesis, an improved algorithm for computing the 2-descent of Case A (i.e there is only gauge equivalence involved between L and $\sigma(L)$) is also proposed. This improved algorithm avoids computing the linear system especially for a complex differential operator. Also, this improved algorithm works better when we apply our 2-descent algorithms to the higher order differential operator.

6.2 Future Work

Second order differential equations with ${}_2F_1$ solutions exist widely in Physics, Combinatorics. Unfortunately, these solutions can not be found by the existing computer system. This situation makes the work presented in this thesis meaningful and valuable. Also, if, after 2-descent, $\tilde{L}$ turns out to have 4 true singularities instead of 3. we can still find its ${}_2F_1$ solutions which will be one direction of future work.

At the moment, we only consider σ's in 2-descent that are defined over the same field of constants C over which L is defined. We can modify the Compute Möbius transformations algorithm to also find σ's defined over an extension of C. However, for such σ we do not plan to compute 2-descent because if there exists descent w.r.t. a σ that is not defined over C, then a larger descent should exist as well, like $C_2 \times C_2$, D_n, A_4, S_4, or A_5, descent.

Another future direction of this work is on finding (if it exists) descent to subfields of index 3. Degree 3 extensions need not be Galois, and so in general, to find 3-descent it is not enough to try all Möbius transformations that fix the singularity structure.

APPENDIX A

MAPLE COMMANDS

All the algorithms in this thesis are implemented in Maple. Therefore, this appendix will demonstrate the most commonly used commands in this thesis.

```
>With(DEtools):
```

This command downloads the DEtools package. This package includes all the commands related to the differential equations solving. By using this command, you can use any command by just typing the command name. Otherwise, without With(DEtools) command, you have to use the long version of the command DEtools[command name]

```
>de2diffop(equation,y(x)); diffop2de(operator, y(x));
```

The de2diffop(equation, $y(x)$) command converts a differential equation to a differential operator. On the other hand, diffop2de(operator, $y(x)$) converts a differential operator to its corresponding differential equation in $y(x)$.

```
>gen_exp(L,T,x=a)
```

This command computes the generalized exponents for L at the point $x = a$.

Example 18. *let* $L := (x-1)\partial^2 + \partial$

```
>gen_exp(L,T,x=infinity)
```

$$[[0, 0, T = \frac{1}{x}]]$$

We can see from this result, we have two exponents $0, 0$ *at* $x = \infty$ *for* L.

```
>LCLM(L_1, L_2);
```

This command computes the Least Common Left Multiple of L_1 and L_2, which is the operator with the minimal order such that all solutions of L_1 and L_2 are solutions of LCLM(L_1,L_2).

```
>symmetric_product(L_1,L_2);
```

This command computes a differential operator which is the homomorphic image of the tensor product $L_1 \circledS L_2$. The result of this command is a linear differential operator of minimal order such that for every solution y_1 of L_1, y_2 of L_2, the product $y_1 \times y_2$ is a solution of it.

```
>Homomorphisms(L_1,L_2);
```

This command computes the homomorphisms between the solution spaces $V(L_1)$ and $V(L_2)$. Assume this homomorphisms is h, then $h(V(L_1))$ would be a subset of $V(L_2)$.

This command differs the **equiv** program because equiv computes an equivalence between $V(L_1)$ and $V(L_2)$ and only for order 2 differential operators. However, Homomorphisms computes a one way map and for any order.

Example 19. *Let* $L_1 = \partial^3$*,* $L_2 := Dx^2$*;*

```
>Homomorphisms(L_1, L_2);
```

$$[\partial, x\partial - 2, \partial^2, x\partial^2 - \partial, x^2\partial^2 - 2x\partial + 2, x^3\partial^2 - 2x^2\partial + 2x]$$

which is a basis of all the operators h*.*

```
>expsols(L, v); expsols(list, g, x);
```

This command computes the exponential solutions of a linear ordinary differential equation. There are two input formats. In the first one expsols(L, v), L is a linear differential equation, and v is the dependent variable of L. In the second format expsols($list$, g, x), $list$ represents the list of coefficients of a linear differential equation, g is the right hand side of the equation, and x is the independent variable of the equation.

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BIOGRAPHICAL SKETCH

Tingting Fang was born on November 13, 1982 in Yichuan, China. In 2004, she completed her Bachelor's degree in Mathematics with emphasis on Mathematics education at Luoyang Normal University, China. After that, she continued her mathematics study and graduated from Southwest Jiaotong University in 2007 with a M.S degree. She entered Florida State University to pursue her Ph.D degree in the same year under the supervision of Dr. Mark. van Hoeij. Tingting's research interests include symbolic computation for differential equations, differential algebra and mathematics education.